$$f(x) = \frac{3}{x^2 - x - 2} = \frac{-1}{x+1} + \frac{1}{x-2}$$

$$= \sum_{n=0}^{\infty} \underbrace{(-1)^{n+1}} x^n + \sum_{n=0}^{\infty} \frac{1}{\underbrace{2^{n+1}}} x^n$$

$$\boxed{f(x) = \sum_{n=0}^{\infty} \left[(-1)^{n+1} + \frac{1}{2^{n+1}} \right] x^n}$$

I.O.C: $\boxed{(-1, 1)}$

$(1-x)^{-1}$

E.g. 4. 1 $f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$; I.O.C. $(-1, 1)$

$g(x) = \frac{1}{(1-x)^2}$ → series centered at 0 for g.

$$f'(x) = -1(1-x)^{-2} \cdot (-1) = (1-x)^{-2} = \frac{1}{(1-x)^2}$$

So, $g(x) = f'(x)$

Series for g: take derivative of series for f

$$g(x) = \sum_{n=1}^{\infty} n x^{n-1}$$

$x = 1$. plug in series: $\sum_{n=1}^{\infty} n \cdot (1)^{n-1} = \sum_{n=1}^{\infty} n$ diverges by n^{th} term test

$x = -1$: $\sum_{n=1}^{\infty} n(-1)^{n-1}$ diverges by n^{th} term test.

Conclusion: $\dfrac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}$; I.O.C $(-1, 1)$

2 $h(x) = \dfrac{x}{(1-x)^2}$

$h(x) = x \cdot \sum_{n=1}^{\infty} n x^{n-1} = \sum_{n=1}^{\infty} n x^n$.

Conclusion: $\boxed{\dfrac{x}{(1-x)^2} = \sum_{n=1}^{\infty} n x^n}$; I.O.C. $(-1, 1)$

Sum of series $\sum_{n=1}^{\infty} \dfrac{n}{2^n}$: Plug $x = \frac{1}{2}$ to both sides of the above equation:

$\boxed{2 = \sum_{n=1}^{\infty} \dfrac{n}{2^n}}$

3) $$u(x) = \frac{2x^2}{(1-x)^3} = 2x^2 \cdot \frac{1}{(1-x)^3}$$

<u>Know:</u> $$\underbrace{\frac{1}{(1-x)^2}} = \underbrace{\sum_{n=1}^{\infty} n x^{n-1}}$$

$\downarrow$ derivative $\qquad\qquad$ $\downarrow$ derivative

$$\frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1)x^{n-2}$$

$\downarrow$ mult by x^2

$$\frac{2x^2}{(1-x)^3} = x^2 \sum_{n=2}^{\infty} n(n-1)x^{n-2}$$

$$\boxed{u(x) = \frac{2x^2}{(1-x)^3} = \underbrace{\sum_{n=2}^{\infty} n(n-1)\boxed{x}^n}}$$ I.O.C. $(-1,1)$

$$\sum_{n=2}^{\infty} \frac{n^2-n}{2^n} = \sum_{n=2}^{\infty} \frac{n(n-1)}{2^n} \cdot \quad \boxed{x = \frac{1}{2}}$$

Wednesday, June 26, 2019 2:55 PM

$$\frac{2\left(\frac{1}{2}\right)^2}{\left(1-\frac{1}{2}\right)^3} = \sum_{n=2}^{\infty} n(n-1)\left(\frac{1}{2}\right)^n$$

$$\frac{1/2}{1/8} = \sum_{n=2}^{\infty} \frac{n(n-1)}{2^n}$$

$$\boxed{4 = \sum_{n=2}^{\infty} \frac{n^2-n}{2^n}}$$

4 So far: $\sum_{n=1}^{\infty} \frac{n}{2^n} = 2$. (Part 2)

$$\sum_{n=1}^{\infty} \frac{n^2-n}{2^n} = 4 \quad \text{(Part 4)}$$

Add these:

$$\sum_{n=1}^{\infty} \left(\frac{n}{2^n} + \frac{n^2-n}{2^n}\right) = 6$$

$$\sum_{n=1}^{\infty} \frac{\cancel{n}+n^2\cancel{-n}}{2^n} = 6$$

$$\boxed{\sum_{n=1}^{\infty} \frac{n^2}{2^n} = 6}$$

Wednesday, June 26, 2019 2:59 PM

E.g.5:

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \quad ; \quad \text{I.O.C. } (-1,1)$$

$$\int \frac{1}{1+x}\,dx = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{n+1}}{n+1} + C$$

$$\boxed{\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C}$$

Plug in $x=0$ to both sides:

$$\ln 1 = C \to C = 0$$

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}.$$

I.O.C. contains: $(-1,1)$

check $x=1$: $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ converges by A.S.T

check $x=-1$: $\displaystyle\sum_{n=0}^{\infty} (-1)^n \cdot \frac{(-1)^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{2n+1}}{n+1}$

$\displaystyle = \sum_{n=0}^{\infty} \frac{-1}{n+1}$ diverges.

$$\boxed{\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \; ; \quad I.O.C. \; (-1,1]}$$

(2) Change of index: Let $n+1 = m$.

$$\ln(1+x) = \sum_{m=1}^{\infty} (-1)^{m-1} \frac{x^m}{m}$$

$$\boxed{\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \; ; \quad I.O.C. \; (-1,1]}$$

E.g.6.

(1) $f(x) = \dfrac{1}{1+x} = \displaystyle\sum_{n=0}^{\infty} (-1)^n x^n \; ; \quad I.O.C. \; (-1, 1)$

$$g(x) = f(x^2) = \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} . \quad I.O.C. \; (-1,1)$$

Integrate Integrate

$$\int \frac{1}{1+x^2} \, dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C$$

$$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{2n+1} + C$$

Plug in $x=0$ to both sides:

$$\arctan(0) = 0 + C \rightarrow C = 0$$

So,

$$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

I.O.C. contains $(-1,1)$

Check $x=1$: $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ converges b/c A.S.T.

check $x=-1$: $\sum_{n=0}^{\infty} (-1)^n \cdot \frac{(-1)^{2n+1}}{2n+1}$.

$$= \sum_{n=0}^{\infty} \frac{\overbrace{(-1)^n \cdot (-1)^n}\cdot (-1)^{n+1}}{2n+1}$$

$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1}$ converges b/c of A.S.T.

Conclusion:

$$\boxed{\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \; ; \text{ I.O.C } [-1,1]}.$$

Wednesday, June 26, 2019 3:15 PM

$x = \frac{1}{\sqrt{3}}$: to both sides of this equation :

$$\underbrace{\arctan\left(\frac{1}{\sqrt{3}}\right)} = \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{\sqrt{3}}\right)^{2n+1}}{2n+1}$$

$$\frac{\pi}{6} = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{\left(\frac{1}{\sqrt{3}}\right)^{2n+1}}{2n+1}$$

$$\pi = 6 \sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{(\sqrt{3})^{2n+1} \cdot (2n+1)}$$

$$\pi = \frac{6}{\sqrt{3}} \sum_{n=0}^{\infty} (-1)^n \frac{1}{3^n (2n+1)}$$

$$\boxed{\pi = 2\sqrt{3} \sum_{n=0}^{\infty} (-1)^n \frac{1}{3^n (2n+1)}}$$