Taylor and Maclaurin Series

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If f has a power series representation:

$$f(x) = \sum_{n=0}^{\infty} \boxed{c_n} (x-\alpha)^n$$

coeff. of series

then:

$$\boxed{c_n = \frac{f^{(n)}(\alpha)}{n!}}$$

Given a function f, then its power series representation is:

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\alpha)}{n!} (x-\alpha)^n}$$

$$f(x) = f(\alpha) + f'(\alpha)(x-\alpha) + \frac{f''(\alpha)}{2!}(x-\alpha)^2 + \frac{f'''(\alpha)}{3!}(x-\alpha)^3 + \ldots$$

This is called the Taylor series for f centered at α.

If the center $\alpha = 0$, then we call it the Maclaurin series for the function.

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n}$$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

* E.g. Construct the Maclaurin series for $f(x) = e^x$.

$$f(0) = 1 \; ; \quad f'(x) = e^x \; ; \quad f^{(n)}(x) = e^x$$

$$f'(0) = 1 \; ; \quad f^{(n)}(0) = 1$$

$$\boxed{e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots}$$

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}}$$

Ratio test: $\left| \frac{x^{\cancel{n+1}}}{(n+1)\cancel{!}} \cdot \frac{\cancel{n!}}{\cancel{x^n}} \right|$

$$\lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1 \rightarrow \text{series converges for all values of } x$$

I.O.C. $(-\infty, \infty)$

E.g. 1. Maclaurin series for $f(x) = \sin x$

$$\sin x = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

$f(x) = \sin x$	$f(0) = 0$
$f'(x) = \cos x$	$f'(0) = 1$
$f''(x) = -\sin x$	$f''(0) = 0$
$f'''(x) = -\cos x$	$f'''(0) = -1$
$f^{(4)}(x) = \sin x$	$f^{(4)}(0) = 0$

$$\boxed{\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots}$$

$$\boxed{\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}}$$ I.O.C. $(-\infty, \infty)$

take deri.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

I.O.C. $(-\infty, \infty)$

$$\boxed{\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}}$$

I.O.C.

$(-\infty, \infty)$

E.g. 2.

(1) $g(x) = e^{\boxed{-3x}}$

$$f(x) = e^{x} = \sum_{n=0}^{\infty} \frac{\boxed{x}^{n}}{n!}$$

$$g(x) = f(-3x) = \sum_{n=0}^{\infty} \frac{(-3x)^{n}}{n!} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^{n} 3^{n} x^{n}}{n!}}$$

(2) $h(x) = \ln(1+x^2)$

$$f(x) = \ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{n}}{n}$$

$$h(x) = \ln(1+x^2) = f(x^2) = \boxed{\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n}}{n}}$$

(3) $$\cos(\pi x) = \sum_{n=0}^{\infty} (-1)^{n} \frac{(\pi x)^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} (-1)^{n} \frac{\pi^{2n} x^{2n}}{(2n)!}$$

(4) $v(x) = 2\sin(x^3)$

$$= 2 \sum_{n=0}^{\infty} (-1)^n \frac{(x^3)^{2n+1}}{(2n+1)!}$$

$$= \boxed{2 \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3}}{(2n+1)!}}$$

(5) $w(x) = x\cos\left(\frac{1}{2}x^2\right)$

$$= x \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{2}x^2\right)^{2n}}{(2n)!}$$

$$= x \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{4^n (2n)!}$$

$$= \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{4^n (2n)!}}$$

(6) $z(x) = x^2 \arctan(x^3)$

$$= x^2 \sum_{n=0}^{\infty} (-1)^n \frac{(x^3)^{2n+1}}{2n+1}$$

$$= x^2 \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3}}{2n+1}$$

$$= \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+5}}{2n+1}}$$

E.g. 3.

$$g(x) = \sin^2 x. = \frac{1 - \cos(2x)}{2}$$

$$= \frac{1}{2}\left(1 - \boxed{\cos(2x)}\right)$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots$$

$$1 - \cos(2x) = \cancel{-1} \cancel{1} + \frac{2^2 x^2}{2!} - \frac{2^4 x^4}{4!} + \frac{2^6 x^6}{6!} - \frac{2^8 x^8}{8!} + \dots$$

$$\underbrace{\frac{1}{2}\left(1 - \cos(2x)\right)}_{\sin^2 x} = \frac{2x^2}{2!} - \frac{2^3 x^4}{4!} + \frac{2^5 x^6}{6!} - \frac{2^7 x^8}{8!} + \dots$$

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$$h(x) = \frac{1}{2}\left(e^{x} + e^{-x}\right)$$

$$e^{x} = 1 \cancel{+ x} + \frac{x^2}{2!} \cancel{+ \frac{x^3}{3!}} + \frac{x^4}{4!} \cancel{+ \frac{x^5}{5!}} + \cdots$$

$+$

$$e^{-x} = 1 \cancel{- x} + \frac{x^2}{2!} \cancel{- \frac{x^3}{3!}} + \frac{x^4}{4!} \cancel{- \frac{x^5}{5!}} + \cdots$$

$$e^{x} + e^{-x} = 2 + 2\cdot\frac{x^2}{2!} + 2\cdot\frac{x^4}{4!} + \cdots$$

$$h(x)\ \frac{1}{2}\left(e^{x} + e^{-x}\right) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

$$\boxed{h(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}}$$

E.g. 4. Find the first 4 nonzero terms of the Maclaurin series for $g(x) = e^x \cdot \cos x$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

$$- \frac{x^2}{2!} - \frac{x^3}{2!} - \frac{x^4}{2!\,2!}$$

$$+ \qquad \frac{x^4}{4!}$$

$$1 + x + 0 - \frac{x^3}{3} - \frac{x^4}{6} \cdots$$

$$\boxed{e^x \cos x = 1 + x - \frac{x^3}{3} - \frac{x^4}{6} \cdots}$$

E.g.5. $\int_0^1 e^{-x^2}\,dx$ Estimate to an error of 0.001

$\frac{x^5}{5!}$

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots$$

$$\int_0^1 e^{-x^2}\,dx = \int_0^1 \left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \frac{x^{10}}{5!} + \cdots\right.$$

$$= \left(x - \frac{x^3}{3} + \frac{x^5}{5(2!)} - \frac{x^7}{7(3!)} + \frac{x^9}{9(4!)} - \frac{x^{11}}{11(5!)} + \cdots\right)\Big|_0^1$$

$$\int_0^1 e^{-x^2}\,dx = 1 - \frac{1}{3} + \frac{1}{5(2!)} - \frac{1}{7(3!)} + \frac{1}{9(4!)} - \frac{1}{11(5!)}$$

If we use the first 5 terms, error $\leq \frac{1}{11(5!)} \approx 0.00075 \rightarrow$ good!

Estimate: $1 - \frac{1}{3} + \frac{1}{5(2!)} - \frac{1}{7(3!)} + \frac{1}{9(4!)} = \leftarrow$ calculator

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Binomial series

$$(1+x)^k = \sum_{n=0}^{\infty} \frac{k(k-1)(k-n+1)x^n}{n!}$$

$$= 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \dots$$

E.g. 6.

$\rightarrow k$

① $f(x) = \sqrt[4]{1+x} = (1+x)^{\frac{1}{4}}$

$$(1+x)^{\frac{1}{4}} = 1 + \frac{1}{4}x + \frac{\frac{1}{4}\left(\frac{1}{4}-1\right)}{2!}x^2 + \frac{\frac{1}{4}\left(\frac{1}{4}-1\right)\left(\frac{1}{4}-2\right)}{3!}x^3 + \dots$$

$$= 1 + \frac{1}{4}x - \frac{3}{32}x^2 + \frac{21}{384}x^3 - \dots$$

② $g(x) = \dfrac{1}{\sqrt{4-x^2}} = (4-x^2)^{-\frac{1}{2}}$

$$= \left(4\left(1-\frac{x^2}{4}\right)\right)^{-\frac{1}{2}} = (4)^{-\frac{1}{2}}\left(1-\frac{x^2}{4}\right)^{-\frac{1}{2}}$$

$$= \frac{1}{2}\left(1-\frac{x^2}{4}\right)^{-1/2}$$

$$= \frac{1}{2}\left[1 + \left(-\frac{1}{2}\right)\left(-\frac{x^2}{4}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}\left(-\frac{x^2}{4}\right)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{3!}\left(-\frac{x^2}{4}\right)^3 + \cdots\right]$$

$\longrightarrow$ Simplify!