Calculus with Parametric Curves

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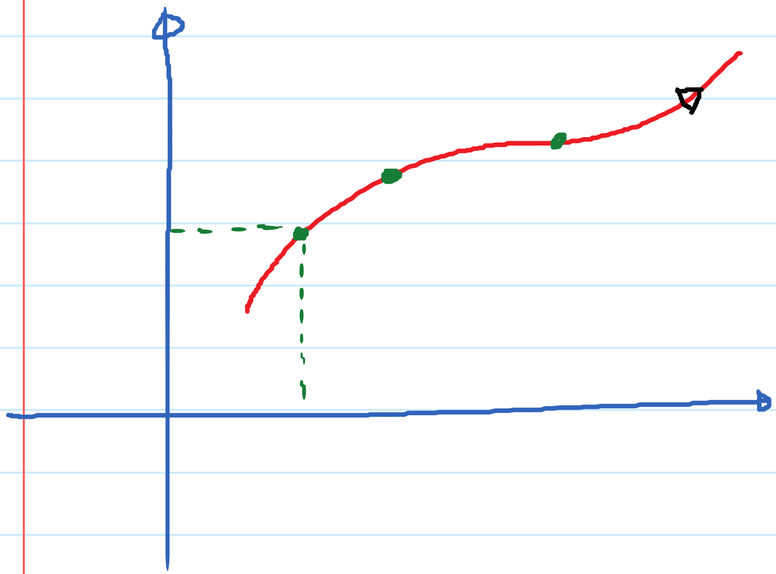


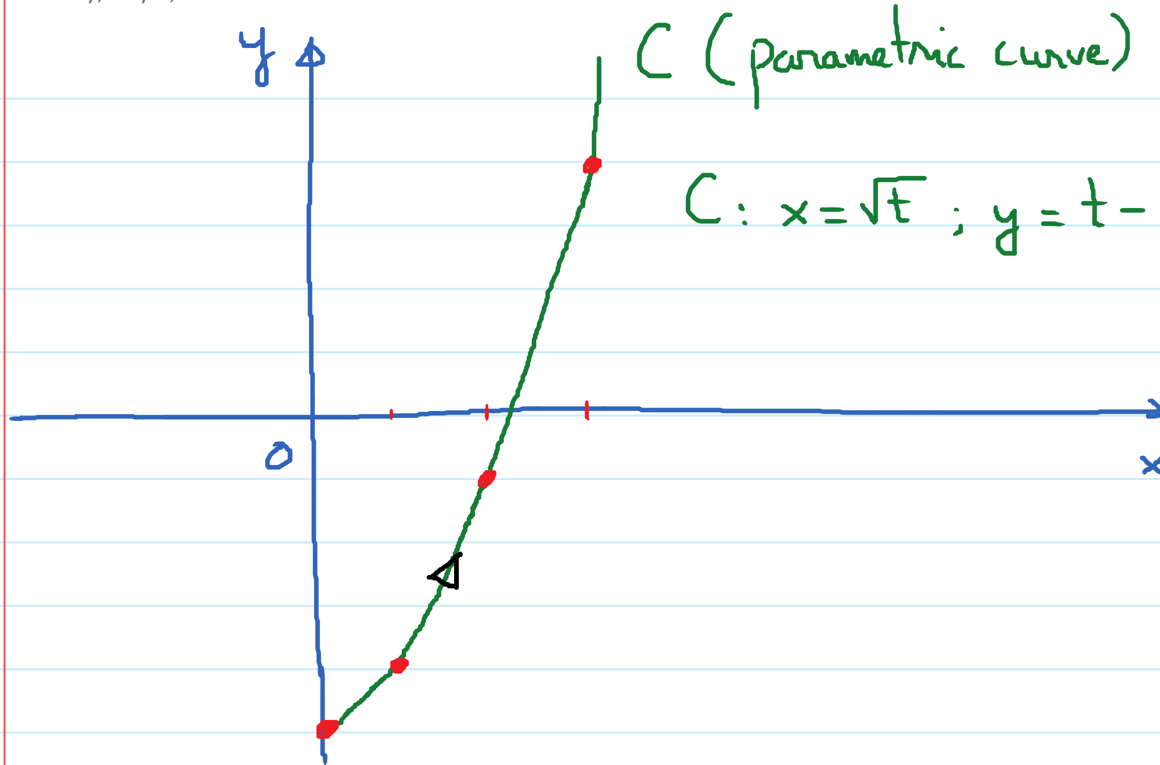
Parametric equations are equations that define both x and y as functions of a third variable t.

$$x = f(t) \quad ; \quad y = g(t).$$

E.g 1

$$x = \sqrt{t} \quad ; \quad y = t - 5 \quad ; \quad t \geq 0$$

t	$x = \sqrt{t}$	$y = t - 5$	Point
0	0	-5	$(0, -5)$
1	1	-4	$(1, -4)$
4	2	-1	$(2, -1)$
9	3	4	$(3, 4)$



$C: x = \sqrt{t}\ ;\ y = t - 5\ ;\ t \geq 0$

Elimata parameter:

$x = \sqrt{\boxed{t}}\ ;\ \boxed{y = t - 5} \longrightarrow \boxed{t = y + 5}$

$x = \sqrt{y+5} \longrightarrow x^2 = y + 5 \longrightarrow y = x^2 - 5;$

$x \geq 0$

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Eg.2. Using trigonometry to eliminate parameter.

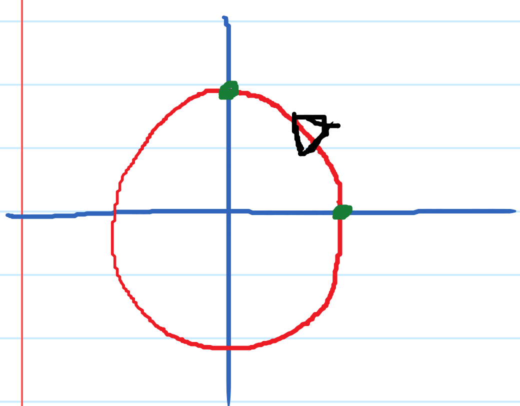
1 $x = \cos(t)$; $y = \sin(t)$; $0 \le t \le 2\pi$.

$$x^2 = \cos^2(t)$$

$$+ \quad y^2 = \sin^2(t)$$

$$x^2 + y^2 = \cos^2(t) + \sin^2(t) = 1$$

$\rightarrow x^2 + y^2 = 1 \rightarrow$ circle centered at $(0,0)$; radius 1.



2 $x = -3 + 4\cos(t)$ $\quad 0 \le t \le 2\pi$.

$y = 2 + 5\sin(t)$

$$\frac{x+3}{4} = \cos(t) \longrightarrow \frac{(x+3)^2}{16} = \cos^2(t)$$

$$\frac{y-2}{5} = \sin(t) \longrightarrow \frac{(y-2)^2}{25} = \sin^2(t)$$

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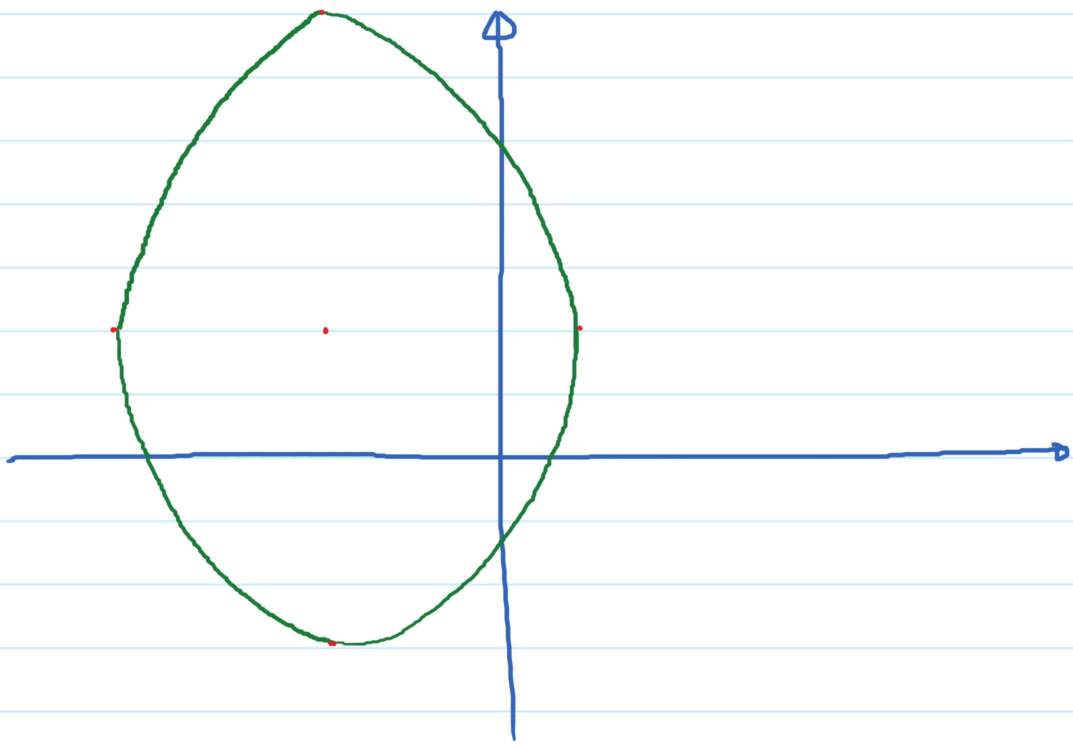
$$\frac{(x+3)^2}{16} + \frac{(y-2)^2}{25} = \cos^2(t) + \sin^2(t) = 1.$$

$$\boxed{\frac{(x+3)^2}{16} + \frac{(y-2)^2}{25} = 1}$$



Find derivatives:

Given a set of parametric equations

$$x = f(t) \; ; \; y = g(t)$$

Find $\frac{dy}{dx}$.

$$\boxed{\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)}}$$

E.g. 3.

(1) $x = \boxed{\sqrt[3]{t}}$; $y = \boxed{4-t}$.

$f(t)$ $g(t)$

$$\frac{dy}{dx} = \frac{g'(t)}{f'(t)} = \frac{-1}{\frac{1}{3}t^{-2/3}} = \boxed{-3t^{2/3}}$$

(2) $x = 2e^{\theta}$; $y = e^{-\theta/2}$

$$\frac{dy}{dx} = \frac{-\frac{1}{2}e^{-\theta/2}}{2e^{\theta}} = \boxed{-\frac{1}{4}e^{-3\theta/2}}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}$$

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1) $x = t^2 + 5t + 4$; $y = 4t$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \boxed{\frac{4}{2t+5}} \leftarrow 1^{st} \text{ derivative.}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \frac{\frac{d}{dt}\left(\frac{4}{2t+5}\right)}{2t+5} = \frac{\frac{-4\cdot(2)}{(2t+5)^2}}{2t+5}$$

$$= \frac{\frac{-8}{(2t+5)^2}}{2t+5} = -\frac{8}{(2t+5)^2}\cdot\frac{1}{(2t+5)} = \boxed{-\frac{8}{(2t+5)^3}}$$

At $t = 0$:

Slope of tangent line: $\left.\frac{dy}{dx}\right|_{t=0} = \frac{4}{2(0)+5} = \boxed{\frac{4}{5}}$

Concavity: $\left.\frac{d^2y}{dx^2}\right|_{t=0} = -\frac{8}{(2(0)+5)^3} = -\frac{8}{125}$

When $t = 0$, the curve is concave down.

Equation of tangent line: Slope $= \boxed{\frac{4}{5}}$, Point:

$x = (0)^2 + 5(0) + 4 = 4$; $y = 4(0) = 0 \rightarrow \boxed{(4,0)}$

Eqn: $y - 0 = \frac{4}{5}(x-4)$

$$\boxed{y = \frac{4}{5}x - \frac{16}{5}}$$

(2) $x = \theta - \sin\theta$; $y = 1 - \cos\theta$; $\theta = \pi$

$\frac{dy}{dx} = \frac{\sin\theta}{1-\cos\theta}$; Slope when $\theta = \pi$: 0

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}\left(\frac{\sin\theta}{1-\cos\theta}\right)}{1-\cos\theta} = \frac{\frac{\cos\theta(1-\cos\theta) - \sin\theta(\sin\theta)}{(1-\cos\theta)^2}}{1-\cos\theta}$$

$$= \frac{\cos\theta - \cos^2\theta - \sin^2\theta}{(1-\cos\theta)^3}$$

$$= \frac{\cos\theta - 1}{(1-\cos\theta)^3} = -\frac{1-\cos\theta}{(1-\cos\theta)^3}$$

$$= -\frac{1}{(1-\cos\theta)^2}$$; when $\theta = \pi$: $\frac{-1}{(-2)^2} = -\frac{1}{4}$

concave down.

Eqn of tangent line : Slope $= 0$

Point : $\theta = \pi$: $x = \pi$; $y = 2$

Equation : $\boxed{y = 2}$