Area and Arc length in Polar coordinates

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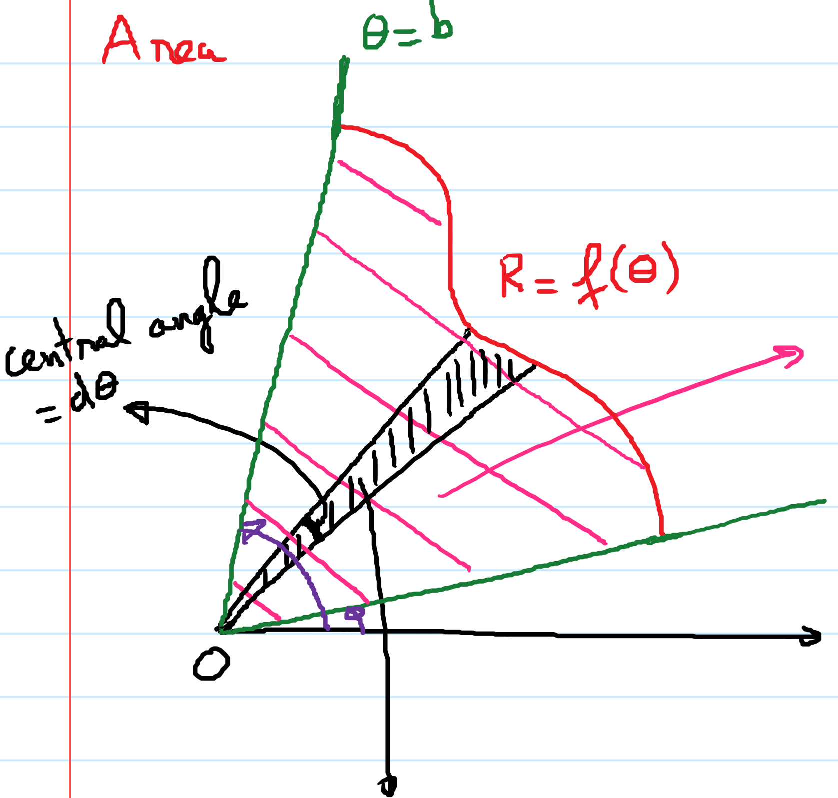
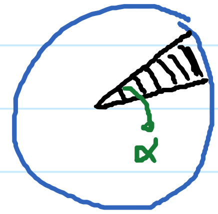
Area





area of a sector of a circle

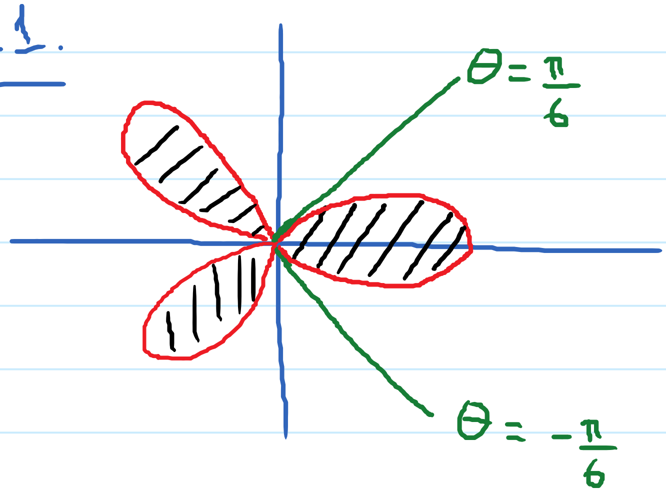
$\frac{1}{2}(\text{Radius})^2 \cdot (\text{central angle})$

$$A_{\text{sector}} = \frac{1}{2} R^2 \, d\theta$$

$$\text{Total area} = \frac{1}{2}\int_a^b R^2 \, d\theta = \boxed{\frac{1}{2}\int_a^b [f(\theta)]^2 \, d\theta}$$

E.g. 1.

$R = 3\cos(3\theta)$



$$A = \frac{1}{2} \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \left[3\cos(3\theta)\right]^2 d\theta$$

$$= \frac{9}{2} \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos^2(3\theta)\, d\theta .$$

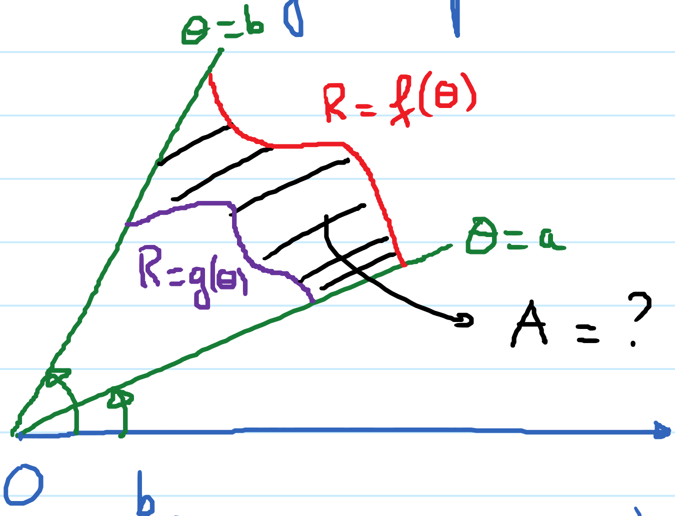
$$= \frac{9}{2} \int_{-\frac{\pi}{6}}^{\pi/6} \frac{1 + \cos(6\theta)}{2}\, d\theta$$

$$= \frac{9}{4} \left(\theta + \frac{\sin(6\theta)}{6} \right) \Bigg|_{-\pi/6}^{\pi/6}$$

$$= \frac{9}{4}\left(\frac{\pi}{6}\right) - \frac{9}{4}\left(-\frac{\pi}{6}\right)$$

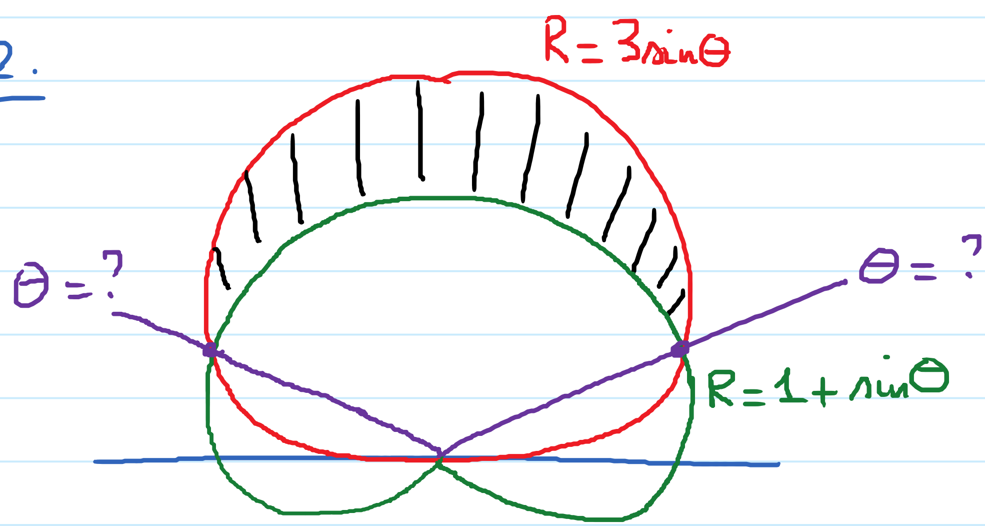
$$= \boxed{\frac{3\pi}{4}}$$

Area bounded by 2 polar curves:



$$A = \frac{1}{2}\int_a^b \left([f(\theta)]^2 - [g(\theta)]^2 \right) d\theta$$

E.g. 2.



Points of intersection: $3\sin\theta = 1 + \sin\theta$.

$$2\sin\theta = 1 \rightarrow \sin\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{6} \; ; \quad \theta = \frac{5\pi}{6}.$$

$$A = \frac{1}{2}\int_{\pi/6}^{5\pi/6}\left[(3\sin\theta)^2 - (1+\sin\theta)^2\right]d\theta$$

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6}\left(9\sin^2\theta - (1 + 2\sin\theta + \sin^2\theta)\right)d\theta$$

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6}\left(8\sin^2\theta - 2\sin\theta - 1\right)d\theta$$

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6}\left(8\cdot\frac{1-\cos(2\theta)}{2} - 2\sin\theta - 1\right)d\theta$$

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6}\left(3 - 4\cos(2\theta) - 2\sin\theta\right)d\theta$$

$$= \frac{1}{2}\cdot\left(3\theta - 2\sin(2\theta) + 2\cos\theta\right)\Big|_{\pi/6}^{5\pi/6}$$

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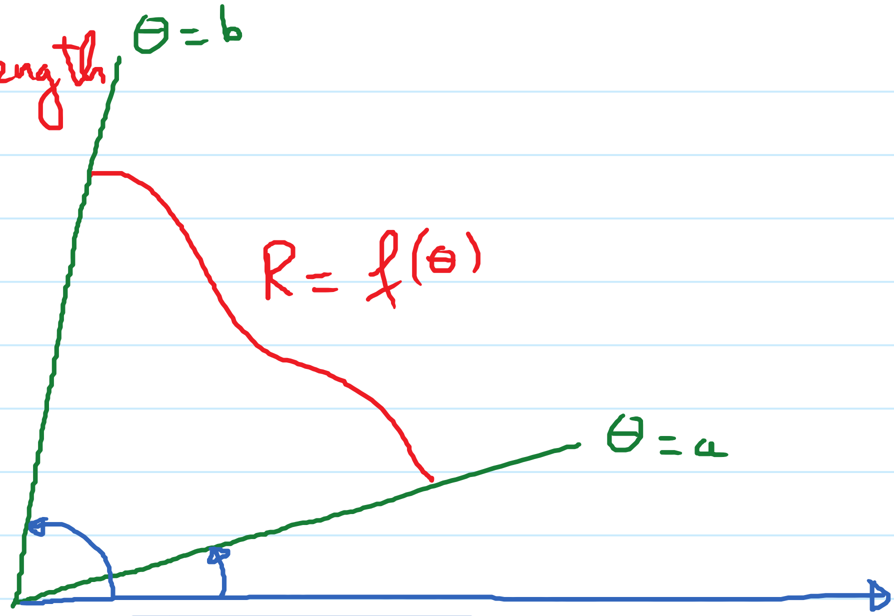
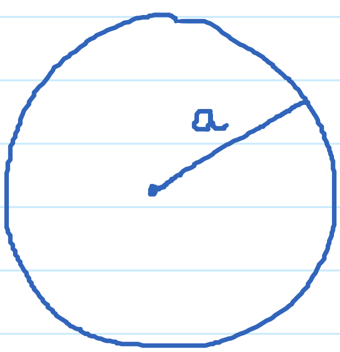
Arc Length



$$\boxed{L = \int_a^b \sqrt{R^2 + \left(\frac{dR}{d\theta}\right)^2}\, d\theta.}$$

E.g. 3. (1) $R = a$; $0 \leq \theta \leq 2\pi$.



$$L = \int_0^{2\pi} \sqrt{a^2 + 0^2}\, d\theta$$

$$= \int_0^{2\pi} a\, d\theta = a \int_0^{2\pi} d\theta$$

$$= a \cdot \theta \Big|_0^{2\pi} = \boxed{2\pi a}.$$

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(2) $R = 1 + \cos\theta \; ; \quad 0 \leq \theta \leq \pi .$

$$L = \int_0^{\pi} \sqrt{R^2 + \left(\frac{dR}{d\theta}\right)^2} \, d\theta$$

$$\frac{dR}{d\theta} = -\sin\theta$$

$$L = \int_0^{\pi} \sqrt{(1 + \cos\theta)^2 + (-\sin\theta)^2} \, d\theta$$

$$L = \int_0^{\pi} \sqrt{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta} \, d\theta$$

$$L = \int_0^{\pi} \sqrt{2 + 2\cos\theta} \, d\theta$$

$$\boxed{\frac{1 + \cos\theta}{2} = \cos^2\left(\frac{\theta}{2}\right)}$$

$$L = \int_0^{\pi} \sqrt{2(1 + \cos\theta)} \, d\theta$$

$$L = \int_0^{\pi} \sqrt{2 \cdot 2\cos^2\left(\frac{\theta}{2}\right)} \, d\theta = \int_0^{\pi} 2\cos\left(\frac{\theta}{2}\right) d\theta .$$

$$2\,\frac{\sin\left(\frac{\theta}{2}\right)}{\frac{1}{2}}\Bigg|^{\pi}$$

$$4\sin\left(\frac{\theta}{2}\right)\Bigg|_{0}^{\pi} = 4\sin\left(\frac{\pi}{2}\right) - 4\sin(0)$$

$$= \boxed{4}$$