Trig Integrals

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Integrals of the form $\int \sin^m(x)\cos^n(x)\,dx$

Case 1: Power of sine is odd

E.g. $\int \sin^3(x)\cos^2(x)\,dx$

Step 1: Rewrite: $\int \boxed{\sin^2(x)}\cos^2(x)\sin(x)\,dx$

Reminder: $\sin^2 x + \cos^2 x = 1$

$\sin^2 x = 1 - \cos^2 x$

$\cos^2 x = 1 - \sin^2 x$

$$= -\int (1-\cos^2 x)\cos^2 x\,\underbrace{(-\sin x\,dx)}_{du}$$

Step 2: u-sub. Let $\boxed{u = \cos x}$. Then $du = -\sin x\,dx$

$$= -\int (1-u^2)u^2\,du$$

Step 3:

$$= -\int (u^2 - u^4)\,du = -\frac{u^3}{3} + \frac{u^5}{5} + C$$

$$= \boxed{-\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C}$$

Case 2: Power of cosine is odd

E.g. $\int \sin^2(\pi x)\cos^5(\pi x)\,dx$.

Step 1: Rewrite: $\int \sin^2(\pi x)\cos^4(\pi x)\cos(\pi x)\,dx$.

$$= \int \sin^2(\pi x)\cdot\left(1-\sin^2(\pi x)\right)^2 \boxed{\cos(\pi x)\,dx} \rightarrow \frac{du}{\pi}$$

Step 2: let $u = \sin(\pi x)$. Then $du = \pi\cdot\cos(\pi x)\,dx$

$$= \frac{1}{\pi}\int u^2\left(1-u^2\right)^2 du$$

Step 3: $= \frac{1}{\pi}\cdot\int u^2\cdot\left(1-2u^2+u^4\right)du$

$$= \frac{1}{\pi}\int\left(u^2-2u^4+u^6\right)du$$

$$= \frac{1}{\pi}\left(\frac{u^3}{3}-\frac{2u^5}{5}+\frac{u^7}{7}\right)+C$$

Then replace u by $\sin(\pi x)$.

Case 3: Both the power of cosine and the power of sine are even.

Reminder:

Double angle formula for cosine

$$\boxed{\begin{aligned}\cos(2x) &= \cos^2(x) - \sin^2(x) = 1 - 2\sin^2(x) \\ &= 2\cos^2(x) - 1\end{aligned}}$$

$$\cos(2x) = 1 - 2\sin^2(x) \longrightarrow \boxed{\sin^2(x) = \frac{1 - \cos(2x)}{2}}$$

$$\cos(2x) = 2\cos^2(x) - 1 \longrightarrow \boxed{\cos^2(x) = \frac{1 + \cos(2x)}{2}}$$

E.g 3. $\int \sin^2 x \cos^2 x \, dx$

$$= \int \frac{1 - \cos(2x)}{2} \cdot \frac{1 + \cos(2x)}{2} \, dx$$

$$= \frac{1}{4} \int (1 - \cos(2x))(1 + \cos(2x)) \, dx$$

$$\frac{1+\cos(4x)}{2}$$

$$= \frac{1}{4}\int \left(1 - \boxed{\cos^2(2x)}\right) dx$$

$$= \frac{1}{4}\int \left(1 - \frac{1+\cos(4x)}{2}\right) dx$$

$$= \frac{1}{4}\int \left(\frac{1}{2} - \frac{1}{2}\cos(4x)\right) dx$$

Note: $\int \cos(Kx)\,dx = \frac{\sin(Kx)}{K}$

$$= \boxed{\frac{1}{4}\left(\frac{x}{2} - \frac{\sin(4x)}{8}\right) + C}$$

2nd way: $\int \sin^2 x \cos^2 x \, dx$

$$= \int \sin^2 x \cdot (1 - \sin^2 x)\, dx = \int (\sin^2 x - \sin^4 x)\, dx$$

$$= \underbrace{\int \sin^2 x \, dx - \int \sin^4 x \, dx}_{\text{done these before! (Previous lecture)}}$$

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Integrals of the form $\int \sec^m(x)\tan^n(x)\,dx$

Reminder: $\boxed{\sec^2 x - \tan^2 x = 1 \rightarrow \begin{cases} \sec^2 x = 1 + \tan^2 x \\ \tan^2 x = \sec^2 x - 1 \end{cases}}$

Case 1: Power of secant is even.

E.g. 4. $\int \tan^5(2x)\sec^4(2x)\,dx$

$$= \int \tan^5(2x)\cdot\sec^2(2x)\cdot\sec^2(2x)\,dx$$

$$= \int \tan^5(2x)\cdot\left(1+\tan^2(2x)\right)\cdot\underbrace{\boxed{\sec^2(2x)\,dx}}_{\frac{du}{2}}.$$

Let $\boxed{u = \tan(2x)}$. Then $du = 2\sec^2(2x)\,dx$

$$= \frac{1}{2}\int u^5\left(1+u^2\right)du = \frac{1}{2}\int\left(u^5+u^7\right)du$$

$$= \frac{1}{2}\left(\frac{u^6}{6}+\frac{u^8}{8}\right)+C = \boxed{\frac{1}{2}\left(\frac{\tan^6(2x)}{6}+\frac{\tan^8(2x)}{8}\right)+C}$$

Case 2: Power of tangent is odd.

E.g. 5: $\int \tan^3(2x)\sec^3(2x)\,dx$

$$= \int \tan^2(2x)\sec^2(2x)\,\boxed{\tan(2x)\sec(2x)\,dx}$$

$$= \int \left(\sec^2(2x)-1\right)\cdot\sec^2(2x)\,\boxed{\tan(2x)\sec(2x)\,dx} \quad \frac{du}{2}$$

Let $\boxed{u = \sec(2x)}$ Then $du = \tan(2x)\sec(2x)\cdot 2\,dx$

$$= \frac{1}{2}\int (u^2-1)\,u^2\,du = \frac{1}{2}\int (u^4-u^2)\,du$$

$$= \frac{1}{2}\left(\frac{u^5}{5}-\frac{u^3}{3}\right)+C$$

$$= \boxed{\frac{1}{2}\left(\frac{\sec^5(2x)}{5}-\frac{\sec^3(2x)}{3}\right)+C}.$$

Neither case 1 nor case 2

E.g. 6 $\int \tan^6 x \, dx$

$$\boxed{\int \tan^n(x)\,dx = \frac{1}{n-1}\tan^{n-1}(x) - \int \tan^{n-2}(x)\,dx}$$

$$\int \tan^6 x \, dx = \frac{1}{5}\tan^5(x) - \int \tan^4(x)\,dx$$

$$= \frac{1}{5}\tan^5(x) - \left[\frac{1}{3}\tan^3(x) - \int \tan^2(x)\,dx\right]$$

$$= \frac{1}{5}\tan^5(x) - \frac{1}{3}\tan^3(x) + \int \tan^2(x)\,dx$$

$$= \frac{1}{5}\tan^5(x) - \frac{1}{3}\tan^3(x) + \tan(x) - \int 1\,dx$$

$$= \boxed{\frac{1}{5}\tan^5(x) - \frac{1}{3}\tan^3(x) + \tan x - x + C}$$

E.g. 7. $\int \sec^3(\pi x)\, \boxed{dx} \rightarrow \frac{du}{\pi}$

$$\boxed{\int \sec^m(x)\,dx = \frac{1}{m-1}\sec^{m-2}(x)\tan(x) + \frac{m-2}{m-1}\int \sec^{m-2}(x)\,dx}$$

Let $u = \pi x$. Then $du = \pi\, dx$

$$\frac{1}{\pi}\int \sec^3(u)\,du = \frac{1}{\pi}\cdot\left[\frac{1}{2}\sec(u)\tan(u) + \frac{1}{2}\int \sec(u)\,du\right]$$

$$= \frac{1}{2\pi}\sec(\pi x)\tan(\pi x) + \frac{1}{2\pi}\ln\left|\sec(\pi x) + \tan(\pi x)\right| + C.$$

Using product to Sum formula.

E.g. 8 $\int \sin(4x)\cos(5x)\,dx = \frac{1}{2}\int\left(\cos(-x) + \sin(9x)\right)dx$

$$= \frac{1}{2}\int\left(\cos x + \sin(9x)\right)dx = \boxed{\frac{1}{2}\cdot\sin x - \frac{\cos(9x)}{18} + C}$$

HW #5

2. $$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n. \end{cases}$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \Big(\cos\big[(m-n)x\big] - \cos\big[(m+n)x\big] \Big)\,dx$$

<u>Case 1:</u> $m \neq n$.

$$= \frac{1}{2}\left(\frac{\sin\big[(m-n)x\big]}{m-n} - \frac{\sin\big[(m+n)x\big]}{m+n} \right)\Bigg|_{-\pi}^{\pi}$$

$$= \frac{1}{2}\left[\left(\frac{\cancelto{0}{\sin\big[(m-n)\pi\big]}}{m-n} - \frac{\cancelto{0}{\sin\big[(m+n)\pi\big]}}{m+n} \right) - \left(\frac{\cancelto{0}{\sin\big[(m-n)(-\pi)\big]}}{m-n} - \frac{\cancelto{0}{\sin\big[(m+n)(-\pi)\big]}}{m+n} \right) \right]$$

$$= 0$$

Case 2: $m = n$.

$$\frac{1}{2}\int_{-\pi}^{\pi}\left(\cos 0 - \cos(2mx)\right)dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi}\left(1 - \cos(2mx)\right)dx$$

$$= \frac{1}{2}\left(x - \frac{\sin(2mx)}{2m}\right)\Bigg|_{-\pi}^{\pi}$$

$$= \frac{1}{2}\left[\left(\pi - \frac{\cancel{\sin(2m\pi)}^{\,0}}{2m}\right) - \left(-\pi - \frac{\cancel{\sin(2m(-\pi))}^{\,0}}{2m}\right)\right]$$

$$= \frac{1}{2}\left[\pi - (-\pi)\right] = \frac{1}{2}\cdot 2\pi = \boxed{\pi}$$