Trigonometric integrals

Key formulas

Some basic trigonometric identities

- Pythagorean identities:
  1. $\sin^2(x) + \cos^2(x) = 1$
     (a) $\sin^2(x) = 1 - \cos^2(x)$
     (b) $\cos^2(x) = 1 - \sin^2(x)$
  2. $\sec^2(x) - \tan^2(x) = 1$
     (a) $\sec^2(x) = 1 + \tan^2(x)$
     (b) $\tan^2(x) = \sec^2(x) - 1$
- Power reduction formulas:
  $$\sin^2(x) = \frac{1-\cos(2x)}{2}; \cos^2(x) = \frac{1+\cos(2x)}{2}$$
- Double angle formulas
  1. $\cos(2x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x) = \cos^2(x) - \sin^2(x)$
  2. $\sin(2x) = 2\sin(x)\cos(x)$
- Product to sum formulas:
  1. $\sin(mx)\sin(nx) = \frac{1}{2}\left(\cos[(m-n)x] - \cos[(m+n)x]\right)$
  2. $\sin(mx)\cos(nx) = \frac{1}{2}\left(\cos[(m-n)x] + \sin[(m+n)x]\right)$
  3. $\cos(mx)\cos(nx) = \frac{1}{2}\left(\cos[(m-n)x] + \cos[(m+n)x]\right)$
- Derivatives of basic trig functions:
  1. $\frac{d}{dx}(\sin(x)) = \cos(x)$
  2. $\frac{d}{dx}(\cos(x)) = -\sin(x)$
  3. $\frac{d}{dx}(\tan(x)) = \sec^2(x)$
  4. $\frac{d}{dx}(\cot(x)) = -\csc^2(x)$
  5. $\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x)$
  6. $\frac{d}{dx}(\csc(x)) = -\csc(x)\cot(x)$
- Antiderivatives of basic trig functions:
  1. $\int \sin(x) = -\cos(x) + C$
  2. $\int \cos(x) = \sin(x) + C$
  3. $\int \tan(x) = \ln|\sec(x)| + C$
  4. $\int \cot(x) = \ln|\sin(x)| + C$
  5. $\int \sec(x) = \ln|\sec(x) + \tan(x)| + C$
  6. $\int \csc(x) = -\ln|\csc(x) + \cot(x)| + C$

Guidelines for trig integrals

- Integrals of the form $\int \sin^m(x)\cos^n(x)dx$
  1. Power of sine, m is odd, i.e., $m = 2k+1$: first rewrite as
  $$\int \sin^{\overbrace{2k+1}^{m}}(x)\cos^n(x)dx = \int \underbrace{\left(\sin^2(x)\right)^k}_{\text{convert to cosine}} \cos^n(x) \overbrace{\sin(x)dx}^{\text{save for } du} = \int \left(1-\cos^2(x)\right)^k \cos^n(x)\sin(x)dx.$$
  Next, let $u = \cos(x)$, so $du = -\sin(x)dx$ and the integral becomes $-\int (1-u^2)^k u^n du$. Then we expand the integrand and integrate.
  2. Power of cosine, n is odd, i.e., $n = 2k+1$: first rewrite as
  $$\int \sin^m(x)\cos^{\overbrace{2k+1}^{n}}(x)dx = \int \sin^m(x) \underbrace{\left(\cos^2(x)\right)^k}_{\text{convert to cosine}} \overbrace{\cos(x)dx}^{\text{save for } du} = \int \sin^m(x)\left(1-\sin^2(x)\right)^k \cos(x)dx.$$
  Next, let $u = \sin(x)$, so $du = \cos(x)dx$ and the integral becomes $\int u^m(1-u^2)^k du$. Then we expand the integrand and integrate.
  3. Powers of both sine and cosine are even. Use the power reduction formula to convert to the second scenario.
- Integrals of the form $\int \sec^m(x)\tan^n(x)dx$
  1. Power of secant, m is even, i.e., $m = 2k$: save a secant-squared factor and rewrite as
  $$\int \sec^{\overbrace{2k}^{m}}(x)\tan^n(x)dx = \int \underbrace{\left(\sec^2(x)\right)^{k-1}}_{\text{convert to tan}} \tan^n(x) \overbrace{\sec^2(x)dx}^{\text{save for } du} = \int \left(1+\tan^2(x)\right)^{k-1} \tan^n(x)\sec^2(x)dx.$$
  Next, let $u = \tan(x)$, so $du = \sec^2(x)dx$ and the integral becomes $\int (1+u^2)^{k-1}u^n du$. Then we expand the integrand and integrate.
  2. Power of tangent, n is odd, i.e., $n = 2k+1$: save a secant-tangent factor and rewrite as
  $$\begin{aligned}\int \sec^m(x)\tan^{\overbrace{2k+1}^{n}}(x)dx &= \int \sec^{m-1}(x) \underbrace{\left(\tan^2(x)\right)^k}_{\text{convert to secant}} \overbrace{\sec(x)\tan(x)dx}^{\text{save for } du} \\ &= \int \sec^{m-1}(x)\left(\sec^2(x)-1\right)^k \sec(x)\tan(x)dx.\end{aligned}$$
  Next, let $u = \sec(x)$, so $du = \sec(x)\tan(x)dx$ and the integral becomes $\int u^{m-1}(u^2-1)^k du$. Then we expand the integrand and integrate.
  3. When there are no secant factors, we can use the reduction formula for tangent to find the integral
  $$\int \tan^n(x)dx = \frac{1}{n-1}\tan^{n-1}(x) - \int \tan^{n-2}(x)dx.$$
  4. When there are no tangent factors, we can use the reduction formula for secant to find the integral
  $$\int \sec^m(x)dx = \frac{1}{m-1}\sec^{m-2}(x)\tan(x) + \frac{m-2}{m-1}\int \sec^{m-2}(x)dx.$$
- Integrals for the form $\int \sin(mx)\sin(nx)dx$, $\int \sin(mx)\cos(nx)dx$, $\int \cos(mx)\cos(nx)dx$: use the product to sum formula.

Example 1: Power of sine is odd

Find the integral $\int \sin^3(x)\cos^2(x)dx$.

Solution

Write the solution here

Example 2: Power of cosine is odd

Find the integral $\int \sin^2(\pi x)\cos^5(\pi x)dx$.

Solution

Write the solution here

Example 3: Powers of sine and cosine are even

Find the integral $\int \sin^2(x)\cos^2(x)dx$.

Solution

Write the solution here

Example 4: Power of secant is even

Find the integral $\int \tan^5(2x)\sec^4(2x)dx$.

Solution

Write the solution here

Example 5: Power of tangent is odd

Find the integral $\int \tan^3(2x)\sec^3(2x)dx$.

Solution

Write the solution here

Example 6: Reduction formula for tangent

Find the integral $\int \tan^6(x)dx$.

Solution

Write the solution here

Example 7: Reduction formula for secant

Find the integral $\int \sec^3(\pi x)dx$.

Solution

Write the solution here

Example 8: Using product to sum formula

Find the integral $\int \sin(4x)\cos(5x)dx$.

Solution

Write the solution here