Review 3

Monday, July 1, 2019 2:08 PM

① $$\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln(n)}} \qquad f(x) = \frac{1}{x\sqrt{\ln(x)}}$$

* $f(x) \geq 0$ on $[2, \infty)$
* Continuous on $[2, \infty)$ (b/c it doesn't include $(-\infty, 1]$)
* Decreasing. $x \cdot \sqrt{\ln(x)}$ is an increasing function so $\frac{1}{x\sqrt{\ln(x)}}$ is a decreasing function on $[2, \infty)$

$$\int_2^{\infty} \frac{1}{x\sqrt{\ln(x)}}\,dx = \lim_{b\to\infty} \int_2^{b} \frac{1}{x\sqrt{\ln(x)}}\,dx$$

du, u

$u = \ln(x)$; $du = \frac{1}{x}\,dx$

$$\int \frac{du}{\sqrt{u}} = \int u^{-\frac{1}{2}}\,du = 2u^{\frac{1}{2}} = 2(\ln(x))^{1/2}$$

$$\lim_{b\to\infty} 2(\ln(x))^{\frac{1}{2}}\Big|_2^b = \lim_{b\to\infty}\left(2(\ln(b))^{\frac{1}{2}} - 2(\ln(2))^{\frac{1}{2}}\right)$$

$$= \infty$$

Series diverges.

(2) $\displaystyle\sum_{n=1}^{\infty} \overbrace{\boxed{\frac{2n^2-1}{3n^5+2n+1}}}^{a_n}$ Limit Comparing this to $\displaystyle\sum_{n=1}^{\infty} \overbrace{\boxed{\frac{1}{n^3}}}^{b_n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2-1}{3n^5+2n+1} \cdot \frac{n^3}{1}$$

$$= \lim_{n\to\infty} \frac{2n^5-n^3}{3n^5+2n+1} = \frac{2}{3} > 0$$

Series behaves like $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges (p-series, $p=3$), the original series converges.

(3) Want: first neglected term ≤ 0.001

$$\frac{1}{2(N+1)^3-1} \leq 0.001 = \frac{1}{1000}$$

$$\to 2(N+1)^3 - 1 \geq 1000$$

$$\to 2(N+1)^3 \geq 1001$$

$$\to (N+1)^3 \geq \frac{1001}{2} \to N+1 \geq \sqrt[3]{\frac{1001}{2}}$$

$$N \geq \sqrt[3]{\frac{1001}{2}} - 1 \approx 6.9\ldots$$

We need at least 7 terms to achieve the required accuracy.

(4) $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2+\cos(n)}{\sqrt{n}}\right|$$

$$0 \leq \left|\frac{2+\cos(n)}{\sqrt{n}}\right| \leq \frac{3}{\sqrt{n}}.$$

As $n \to \infty$; 0

By Squeeze Theorem, $\lim_{n\to\infty}\left|\frac{2+\cos(n)}{\sqrt{n}}\right| = 0 < 1$

So, By the Ratio Test, the series converges.

5 $f(x) = \ln(x)$

$f'(x) = \frac{1}{x}$; $f''(x) = -\frac{1}{x^2}$; $f'''(x) = \frac{2}{x^3}$; $f^{(4)} = -\frac{6}{x^4}$

$f(2) = \ln(2)$; $f'(2) = \frac{1}{2}$; $f''(2) = -\frac{1}{4}$; $f'''(2) = \frac{1}{4}$

$f^{(4)}(2) = -\frac{3}{8}$.

$$T_4(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3 + \frac{f^{(4)}(2)}{4!}(x-2)^4$$

$$\boxed{T_4(x) = \ln(2) + \frac{1}{2}(x-2) - \frac{1}{(2!)(4)}(x-2)^2 + \frac{1}{(3!)4}(x-2)^3 - \frac{3}{(4!)(8)}(x-2)^4}$$

$f(2.1) \approx T_4(2.1) = 0.7$

Monday, July 1, 2019 2:30 PM

6

$$a_n = \frac{n^2 x^n}{2\cdot 4\cdot 6 \cdots (2n)} = \frac{n^2 x^n}{2(\underbrace{1\cdot 2\cdot 3 \cdots n})}$$

$$a_n = \frac{n^2 x^n}{2(n!)}$$

$$a_{n+1} = \frac{(n+1)^2 x^{n+1}}{2[(n+1)!]}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^2 x^{\cancel{n+1}}}{\cancel{2}[(n+1)\cancel{!}]} \cdot \frac{\cancel{2}(\cancel{n!})}{n^2 \cancel{x^n}}\right|$$

$$= \frac{(n+1)^{\cancel{2}}}{\cancel{(n+1)}\cdot n^2}|x| = \frac{n+1}{n^2}|x|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{n+1}{n^2}|x| = 0 < 1$$

Since limit < 1 regardless of values of x, the series converges for all x.

Radius of convergence $= \infty$

I.O.C. $= (-\infty, \infty)$

7 $f(x) = \dfrac{5}{2x-3}$ center at $\alpha = -3$

$$= \frac{5}{-9+2(x+3)}$$

$$= \frac{5}{-9\left(1 - \boxed{\frac{2(x+3)}{9}}\right)} \rightarrow R$$

$$= -\frac{5}{9} \cdot \frac{1}{1 - \frac{2(x+3)}{9}}$$

$$= -\frac{5}{9} \sum_{n=0}^{\infty} R^n$$

$$= -\frac{5}{9} \sum_{n=0}^{\infty} \left(-\frac{2}{9}(x+3)\right)^n$$

$$f(x) = \boxed{-\frac{5}{9} \sum_{n=0}^{\infty} \frac{(-1)^n 2^n (x+3)^n}{9^n}}$$

I.O.C. $|R| < 1 = \left|\dfrac{2(x+3)}{9}\right| < 1$

$$|x+3| < \frac{9}{2} \rightarrow -\frac{9}{2} < x+3 < \frac{9}{2} \rightarrow \boxed{-\frac{15}{2} < x < \frac{3}{2}}$$

Monday, July 1, 2019 2:40 PM

(8)
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \cdots$$

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

$$x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} \cdots$$

$+$

$$- \frac{x^3}{3} - \frac{x^4}{3} \cdots$$

$$e^x \arctan(x) = \boxed{x + x^2 + \frac{x^3}{6} - \frac{x^4}{6}} \cdots$$

(9)
$$\cos(x^2) = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \frac{x^{16}}{8!}$$

$$\int_0^1 \cos(x^2)\,dx = \left(x - \frac{x^5}{5(2!)} + \frac{x^9}{9(4!)} - \frac{x^{13}}{13(6!)} + \frac{x^{17}}{17(8!)}\right)\Bigg|_0^1$$

$$= \underbrace{\boxed{1 - \frac{1}{5(2!)} + \frac{1}{9(4!)} - \frac{1}{13(6!)}}}_{} + \frac{1}{17(8!)}$$

error $\leq \frac{1}{17(8!)}$. Need 4 terms.

Monday, July 1, 2019 2:48 PM

10 $$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad ; \quad \text{I.O.C } (-1,1)$$

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}$$

$$f(x) = \frac{x}{(1-x)^2} = x \sum_{n=1}^{\infty} n x^{n-1} = \sum_{n=1}^{\infty} n x^n$$

Plug $x = \frac{2}{3}$ into both sides.

$$\frac{2/3}{\left(1-\frac{2}{3}\right)^2} = \sum_{n=1}^{\infty} n \cdot \left(\frac{2}{3}\right)^n$$

$$\boxed{6 = \sum_{n=1}^{\infty} n \left(\frac{2}{3}\right)^n}$$