Representation of functions by power series

Key formulas

Basic geometric power series centered at 0:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots = \frac{1}{1-x}, \quad \text{I.O.C.:}(-1,1).$$
$$\sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \dots = \frac{1}{1+x}, \quad \text{I.O.C.:}(-1,1).$$

We say that the function $f(x) = \frac{1}{1-x}$ on the interval (-1,1) is represented by the series $\sum_{n=0}^{\infty} x^n$. Similarly, the function $f(x) = \frac{1}{1+x}$ on the interval (-1,1) is represented by the series $\sum_{n=0}^{\infty} (-1)^n x^n$.

If we know the power series for the functions f(x) and g(x), say $f(x) = \sum_{n=0}^{\infty} c_n x^n$ and $g(x) = \sum_{n=0}^{\infty} d_n x^n$, we can obtain the series for the functions $f(x) \pm g(x)$, f(kx) (k is a constant), $f(x^p)$ (p is a constant) by the following operations

$$f(x) \pm g(x) = \sum_{n=0}^{\infty} (c_n \pm d_n) x^n$$
$$f(kx) = \sum_{n=0}^{\infty} c_n (kx)^n = \sum_{n=0}^{\infty} c_n k^n x^n$$
$$f(x^p) = \sum_{n=0}^{\infty} c_n (x^p)^n = \sum_{n=0}^{\infty} c_n x^{pn}$$

Note that interval of convergence of the series for $f(x) \pm g(x)$ is the intersection of that of the series for f and for g. We can find the interval of convergence of the series for f(kx) and $f(x^p)$ by finding the values of x for which kx and x^p belongs to the interval of convergence of the series for f.

We can also obtain the series for f'(x) and $\int f(x)dx$ by differentiating and integrating the series for f(x). Important series to know from this lecture:

$$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \quad \text{I.O.C.:} [-1,1].$$
$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}, \quad \text{I.O.C.:} (-1,1].$$

E	xaı	mp	le 1	: F	unc	tion	s re	pre	sent	ted	$\mathbf{b}\mathbf{y}$	geo	met	ric	pov	ver	ser	ies											
Fi	nd	a g	eon	netr	ic po	wer	seri	es c	ente	red	at a	for	the	giv	en fi	ınct	ion.	Det	term	ine	the	inte	rval	of o	conv	erge	ence.		
		-			1														2										
	1.	f(<i>x</i>) =	\overline{x}	$\frac{1}{+2}$,	$\alpha =$	0.									2. j	f(x)	$=\frac{1}{6}$	$\frac{z}{-x}$	$, \alpha$	= -	-2.							

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	Exa	mpl	e 2	: Fi	nd	pov	ver	seri	es ı	ısin	g b	asic	ор	era	tion	S															
	T	1		c	C			1	1	(\ \	1		,						с 1	.1						1		c ,	1	
	Use	the f	serie	es ic	r f Det	(x) =	1	-x	and	g(x)) =	1 +	\overline{x}^{a}	nd	serie	s op	erat	lons	to 1	ind	the	ром	ver s	serie	s ce	nter	ed a	t U .	for t	the	
ł	giver	l Iui	ICUI)II . .	Der	erm	ine t	ne i	nter	var		nve	rgen	ice.																	
	1.	u(x)	c) =	1	1	2.										2. ı	y(x)	$=\frac{1}{2}$	$\frac{x}{r^2}$ +	<u> </u>											
				1 -	-4x													20		-											

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Example 3: Find power series	using basic operations	
		0
Find the partial fraction decompo	sition of $f(x) = \frac{1}{x^2 - x - 2}$ and use it to find the power series for f centered at	0.
Determine the interval of converge	nice.	

Solu	itio	n															
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1	Exa	mp	le 4	: Fi	ind	pov	ver	\mathbf{seri}	les l	oy c	liffe	ren	tiat	ion																
	-1.	Us (N	e th ote	e p tha	ower t q(2	$(\sin x) =$	$\operatorname{ies} \mathfrak{c}$	$\left(x^{2}\right) $	ered Det	at erm	0 fo	f(x) =	= <u> </u>	$\frac{1}{-x}$ of c	to f	ind ergei	a po nce (ower of th	seri ne se	ies c ries	ente	ered	at	0 foi	r g(:	r) =	= (1	$\frac{1}{-x}$	$\overline{)^2}$.
	2.	Ús	e th	e se	eries	froi	n P	art	1 to	obt	ain	$_{\mathrm{the}}$	seri	es c	ente	red	at 0	for	h(x	;) =	(1 -	$\frac{x}{-x}$	$\frac{1}{2}$.	Use	this	res	ult	to fi	nd t	the
		su	n o	the	e ser	ies_{r}	$\sum_{n=1}^{\infty} \frac{1}{2}$	$\frac{n}{2^n}$.																						
	3.	Fii	nd a	po	wer	serie	es ce	enter	ed a	at O	for '	u(x)) = -	$\frac{2x}{(1-x)^2}$	$(x)^{2}$. Us	se th	is re	esult	t to	find	the	sur	n of	the	seri	$es \sum_{n}$	$\sum_{n=2}^{\infty} \frac{n}{2}$	$\frac{a^2}{2^n}$	$\frac{n}{-}$.
	4.	Us	e th	e pi	revic	ous r	esul	ts t	o fin	d tł	ne su	ım c	of th	e se	ries	$\sum_{n=1}^{\infty}$	$\frac{n^2}{2^n}.$													

Solu	itio	n															
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Exa	mp]	le 5	: Fi	nd	pov	ver	seri	es l	oy i	nte	grat	ion	L																	
1.	Us	e th	ie se	ries	cen	tere	d at	0	or j	r(x)	=	$\frac{1}{1+}$	$\frac{1}{x}$ to) fin	d a	seri	es c	ente	red	at () for	g(x	c) =	ln((1 +	x)	(Not	e tł	iat	
	g(x)	r) =	Jf	$(x)\epsilon$	lx.)																									
2.	Pe	rfon	ır a	cha	nge	of ir	ndex	to	shov	v th	at li	n(1 -	+x	= `	$\sum_{k=1}^{\infty}$	$(-1)^{r}$	$n - 1 \frac{2}{2}$	$\frac{x^n}{n}$												
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Solu	itio	n															
Writ	e th	e so	lutio	on h	ere												

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1	. Use t	he ser	ies cer	tered a	at 0 for	f(x) =	1	and se	ries op	eratio	ns to fir	nd a sei	ies ce	ntered	at 0	for $a(x)$) = -	1
						J (~)	1 + x	and se	noo op	01000		14 4 50.				101 g (w	' 1	$+x^{2}$
	Then	use t	he ser	ries for	q(x) t	o find a	a serie	s center	red at	0 for	h(x) =	arctar	(x).	(Note	that	h(x) =	f q(x)dx.)
	Deter	rmine	the in	nterval	of conv	rgenc	e of ea	ch serie	es.		< /			`			J 0 (<u> </u>
										∞	$(1)^{\gamma}$	ı						
2	Uso t	ho roc	ult of	the n	rovious	nart to	show	that π	$-2\sqrt{2}$	\overline{X}	(-1)							

Solution																							
	Writ	e the	solı	itio	n h	ere																	