Representation of functions by power series

Key formulas

Basic geometric power series centered at 0:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}, \qquad \text{I.O.C.:}(-1, 1).$$

$$\sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \cdots = \frac{1}{1+x}, \qquad \text{I.O.C.:}(-1, 1).$$

We say that the function $f(x) = \dfrac{1}{1-x}$ on the interval $(-1, 1)$ is represented by the series $\displaystyle\sum_{n=0}^{\infty} x^n$.

Similarly, the function $f(x) = \dfrac{1}{1+x}$ on the interval $(-1, 1)$ is represented by the series $\displaystyle\sum_{n=0}^{\infty} (-1)^n x^n$.

If we know the power series for the functions $f(x)$ and $g(x)$, say $f(x) = \displaystyle\sum_{n=0}^{\infty} c_n x^n$ and $g(x) = \displaystyle\sum_{n=0}^{\infty} d_n x^n$, we can obtain the series for the functions $f(x) \pm g(x)$, $f(kx)$ (k is a constant), $f(x^p)$ (p is a constant) by the following operations

$$f(x) \pm g(x) = \sum_{n=0}^{\infty} (c_n \pm d_n) x^n$$

$$f(kx) = \sum_{n=0}^{\infty} c_n (kx)^n = \sum_{n=0}^{\infty} c_n k^n x^n$$

$$f(x^p) = \sum_{n=0}^{\infty} c_n (x^p)^n = \sum_{n=0}^{\infty} c_n x^{pn}$$

Note that interval of convergence of the series for $f(x) \pm g(x)$ is the intersection of that of the series for f and for g.
We can find the interval of convergence of the series for $f(kx)$ and $f(x^p)$ by finding the values of x for which kx and x^p belongs to the interval of convergence of the series for f.
We can also obtain the series for $f'(x)$ and $\int f(x)dx$ by differentiating and integrating the series for $f(x)$.
Important series to know from this lecture:

$$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \qquad \text{I.O.C.:}[-1, 1].$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}, \qquad \text{I.O.C.:}(-1, 1].$$

Example 1: Functions represented by geometric power series

Find a geometric power series centered at α for the given function. Determine the interval of convergence.

1. $f(x) = \dfrac{1}{x+2}$, $\alpha = 0$.
2. $f(x) = \dfrac{2}{6-x}$, $\alpha = -2$.

Solution

Write the solution here

Example 2: Find power series using basic operations

Use the series for $f(x) = \dfrac{1}{1-x}$ and $g(x) = \dfrac{1}{1+x}$ and series operations to find the power series centered at 0 for the given function. Determine the interval of convergence.

1. $u(x) = \dfrac{1}{1-4x^2}$.
2. $v(x) = \dfrac{x}{2x^2+1}$.

Solution

Write the solution here

Example 3: Find power series using basic operations

Find the partial fraction decomposition of $f(x) = \dfrac{3}{x^2 - x - 2}$ and use it to find the power series for f centered at 0. Determine the interval of convergence.

Solution

Write the solution here

Example 4: Find power series by differentiation

1. Use the power series centered at 0 for $f(x) = \dfrac{1}{1-x}$ to find a power series centered at 0 for $g(x) = \dfrac{1}{(1-x)^2}$. (Note that $g(x) = f'(x)$.) Determine the interval of convergence of the series.
2. Use the series from Part 1 to obtain the series centered at 0 for $h(x) = \dfrac{x}{(1-x)^2}$. Use this result to find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$.
3. Find a power series centered at 0 for $u(x) = \dfrac{2x^2}{(1-x)^3}$. Use this result to find the sum of the series $\displaystyle\sum_{n=2}^{\infty} \frac{n^2 - n}{2^n}$.
4. Use the previous results to find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$.

Solution

Write the solution here

Solution

Example 5: Find power series by integration

1. Use the series centered at 0 for $f(x) = \dfrac{1}{1+x}$ to find a series centered at 0 for $g(x) = \ln(1+x)$ (Note that $g(x) = \int f(x)dx$.)
2. Perfomr a change of index to show that $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \dfrac{x^n}{n}$

Solution

Write the solution here

Example 6: Find power series by integration

1. Use the series centered at 0 for $f(x) = \dfrac{1}{1+x}$ and series operations to find a series centered at 0 for $g(x) = \dfrac{1}{1+x^2}$. Then use the series for $g(x)$ to find a series centered at 0 for $h(x) = \arctan(x)$. (Note that $h(x) = \int g(x)dx$.) Determine the interval of convergence of each series.

2. Use the result of the previous part to show that $\pi = 2\sqrt{3}\sum_{n=0}^{\infty} \dfrac{(-1)^n}{(2n+1)3^n}$.

Solution

Write the solution here