Taylor and Maclaurin Series

Key formulas

If the function f has a power series representation $f(x) = \sum_{n=0}^{\infty} c_n (x-\alpha)^n$ on the interval I, then the coefficients c_n of the series are given by the formula

$$c_n = \frac{f^{(n)}(\alpha)}{n!}.$$

As a result, if f has a power series representation centered at α, it must be of the form

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\alpha)}{n!}(x-\alpha)^n = f(\alpha) + f'(\alpha)(x-\alpha) + \frac{f''(\alpha)}{2!}(x-\alpha)^2 + \frac{f'''(\alpha)}{3!}(x-\alpha)^3 + \ldots.$$

The above series is called the **Taylor series** for f centered at α. If the center $\alpha = 0$, then it is called the **Maclaurin** series for f:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots.$$

List of important Maclaurin series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots \qquad \text{I.O.C: } (-1,1)$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \qquad \text{I.O.C: } (-\infty,\infty)$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \qquad \text{I.O.C: } (-\infty,\infty)$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \qquad \text{I.O.C: } (-\infty,\infty)$$

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots \qquad \text{I.O.C: } [-1,1]$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \qquad \text{I.O.C: } (-1,1]$$

$$(1+x)^k = \sum_{n=0}^{\infty} \frac{k(k-1)\ldots(k-n+1)x^n}{n!} = 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \ldots \qquad \text{I.O.C: } (-1,1)$$

The last series is called the **binomial series** and convergence/divergence at the endpoints ± 1 depends on the values of k.

Example 1: Maclaurin series for $\sin x$ and $\cos x$

Find the Maclaurin series for $f(x) = \sin x$. Use this result and term-by-term differentiation to find the Maclaurin series for $g(x) = \cos x$.

Solution

Write the solution here

Example 2: Using the important Maclaurin series

Find the Maclaurin series for the function using the important series:

1. $g(x) = e^{-3x}$
2. $h(x) = \ln(1 + x^2)$
3. $u(x) = \cos(\pi x)$
4. $v(x) = 2\sin(x^3)$
5. $w(x) = x\cos\left(\frac{1}{2}x^2\right)$
6. $z(x) = x^2\arctan(x^3)$

Solution

Write the solution here

Example 3: Using the important Maclaurin series

Find the Maclaurin series for the function using the important series:

1. $g(x) = \sin^2 x$
2. $h(x) = \frac{1}{2}(e^x + e^{-x})$

Solution

Write the solution here

Example 4: Multiply power series

Find the first four nonzero terms of the Maclaurin series for the function $g(x) = e^x \cos x$.

Solution

Write the solution here

Example 5: Use power series to approximate an integral

Use a power series to approximate $\int_0^1 e^{-x^2} dx$ within an error of 0.001.

Solution

Write the solution here

Example 6: Use the binomial series

Use the binomial series formula to find the Maclaurin series for the given function

1. $f(x) = \sqrt[4]{1+x}$
2. $g(x) = \dfrac{1}{\sqrt{4-x^2}}$

Solution

Write the solution here