Calculus with Parametric Curves

Key formulas

Parametric equations are equations that define x and y as functions of a third variable t (called the **parameter**):

$$x = f(t), y = g(t).$$

Plugging a specific value of t into the equations for x and y determine a point $(x, y) = (f(t), g(t))$ in the xy-plane. As t varies, the point $(x, y) = (f(t), g(t))$ varies and traces out a curve C, called a **parametric curve**.
We can **eliminate the parameter** to find a rectangular equation that represents the graph of a set of parametric equations.
Differentiation: Given a set of parametric equations $x = f(t), y = g(t)$, we have

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)}, \text{ provided } \frac{dx}{dt} \neq 0.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}.$$

The first derivative helps us find slopes of tangent lines to a parametric curve. The second derivative helps us determine concavity of the curve.
Arc Length: If a smooth parametric curve $C : x = x(t), y = y(t), a \leq t \leq b$ does not intersect itself on the interval $a \leq t \leq b$ (except possibly at the endpoints), the the arc length of C on the interval is given by

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2} dt.$$

The area S of the surface of revolution formed by rotating C about the x-axis (assuming $g(t) \geq 0$) is

$$S = 2\pi \int_a^b g(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt,$$

and about the y-axis (assuming $f(t) \geq 0$) is

$$S = 2\pi \int_a^b f(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

Example 1: Sketch a curve described by parametric equations

Sketch the curve describe by the parametric equations, indicate the direction in which the curve is traced as the parameter increases and find the rectangular equation by eliminating the parameter.

$$x = \sqrt{t}, y = t - 5, \text{ where } t \geq 0.$$

Solution

Write the solution here

Example 2: Using trigonometry to eliminate the parameter

Eliminate the parameter to find the rectangular equation for the curve. Identify the curve.

1. $x = \cos(t), y = \sin(t)$ where $0 \leq t \leq 2\pi$.
2. $x = -3 + 4\cos(t), y = 2 + 5\sin(t)$ where $0 \leq t \leq 2\pi$.

Solution

Write the solution here

Example 3: Finding a derivative

Find $\frac{dy}{dx}$ for the curve given by

1. $x = \sqrt[3]{t}$, $y = 4 - t$
2. $x = 2e^{\theta}$, $y = e^{-\theta/2}$

Solution

Write the solution here

Example 4: Find slope and concavity

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ and find the slope, concavity, and equation of the tangent line of the curve at the given point.

1. $x = t^2 + 5t + 4$, $y = 4t$, $t = 0$
2. $x = \theta - \sin\theta$, $y = 1 - \cos\theta$, $\theta = \pi$

Solution

Write the solution here

Example 5: Find arc length

Find the arc length of the curve on the given interval

$$x = \arcsin(t), y = \ln\sqrt{1-t^2}, 0 \leq t \leq 1.$$

Solution

Write the solution here

Example 6: Find surface area

ind the area of the surface generated by revolving the curve $x = a\cos^3\theta$, $y = a\sin^3\theta$, $0 \leq \theta \leq \pi$ about the x-axis.

Solution

Write the solution here