Monday, August 5, 2019 12:02 AM

Properties of Laplace Transforms - I

Translation in s property.

$$\mathcal{L}\{\sin x\} = \frac{1}{s^2+1} \quad ; \quad \mathcal{L}\{e^{2x}\sin x\} = \frac{1}{(s-2)^2+1}$$

$$\mathcal{L}\{\cos x\} = \frac{s}{s^2+1} \quad ; \quad \mathcal{L}\{e^{2x}\cos x\} = \frac{s-2}{(s-2)^2+1}$$

$$\mathcal{L}\{e^{-3x}\cos x\} = \frac{s+3}{(s+3)^2+1}$$

Translation in s property:

$$\boxed{\mathcal{L}\{e^{ax}f(x)\} = F(s-a)}$$

Here $F(s) = \mathcal{L}\{f(x)\}$

Update Table:

Know: $\mathcal{L}\{\underbrace{\boxed{x^n}}_{f(x)}\} = \overbrace{\boxed{\dfrac{n!}{s^{n+1}}}}^{F(s)} \longrightarrow \mathcal{L}\{e^{ax}\cdot \overbrace{\boxed{x^n}}^{f(x)}\} = \overbrace{\boxed{\dfrac{n!}{(s-a)^{n+1}}}}^{F(s-a)}$

$(s>0)$ $\qquad\qquad$ $(s>a)$

$$\mathcal{L}\{\sin(bx)\} = \frac{b}{s^2+b^2} \longrightarrow \mathcal{L}\{e^{ax}\cdot\sin(bx)\} = \frac{b}{(s-a)^2+b^2},$$

$s>a$

$$\mathcal{L}\{\cos(bx)\} = \frac{s}{s^2+b^2} \longrightarrow \mathcal{L}\{e^{ax}\cdot\cos(bx)\} = \frac{s-a}{(s-a)^2+b^2}$$

$s>a$

$$F(s-a) = F(s+2)$$

E.g. 1

(1) $\mathcal{L}\{e^{-2x}\boxed{\cos(4x)}\} = \dfrac{s+2}{(s+2)^2+16}$, with $a = -2$ and $f(x) = \cos(4x)$

$$\mathcal{L}\{f(x)\} = \boxed{\frac{s}{s^2+16}}$$

$F(s)$

(2) $\mathcal{L}\{e^{3x}(9-4x+10\sin(\frac{x}{2})\}$

$$= \mathcal{L}\{e^{3x}\cdot 9\} - 4\mathcal{L}\{e^{3x}\cdot x\} + 10\mathcal{L}\{e^{3x}\cdot\sin(\tfrac{x}{2})\} \quad (\text{linearity})$$

$$= \frac{9}{s-3} - \frac{4}{(s-3)^2} + 10\cdot\frac{\frac{1}{2}}{(s-3)^2+\frac{1}{4}} \quad (\text{Translation in } s \text{ property})$$

Inverse Form of the translation in s property

$$\boxed{\mathcal{L}^{-1}\{F(s-a)\} = e^{ax}\cdot f(x)}$$

where $f(x) = \mathcal{L}^{-1}\{F(s)\}$

E.g. 2

(1) $\mathcal{L}^{-1}\left\{\dfrac{1}{(s-1)^4}\right\}$ $\longrightarrow F(s-1) \longrightarrow F(s) = \dfrac{1}{s^4}$

Now, we find : a $\quad \mathcal{L}^{-1}\left\{\dfrac{1}{s^4}\right\} = \dfrac{1}{3!}\boxed{\mathcal{L}^{-1}\left\{\dfrac{3!}{s^4}\right\}} = \dfrac{x^3}{6}$

$$\longrightarrow \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^4}\right\} = e^{x}\cdot\frac{x^3}{6}$$

(2) $\mathcal{L}^{-1}\left\{\dfrac{2s^2+10s}{(s^2-2s+5)(s+1)}\right\}$

Partial Fractions Decomposition:

$$\frac{2s^2+10s}{\quad} = \frac{As+B}{\quad} + \frac{C}{\quad}$$