Tuesday, August 6, 2019 10:15 AM

E.g. 8

$$y'' + 5y' + 6y = g(x) \quad ; \quad y(0) = 0 \; ; \; y'(0) = 2$$

$$\text{where } g(x) = \begin{cases} 0 & : \; 0 \le x < 1 \\ x & : \; 1 \le x < 5 \\ 1 & : \; x \ge 5 \end{cases}$$

Step 1: Rewrite $g(x)$ as a combination of step functions

$$g(x) = \big(u(x-1) - u(x-5)\big) \cdot x + u(x-5)$$

Step 2: Take laplace transform

$$y'' + 5y' + 6y = \big(u(x-1) - u(x-5)\big) \cdot x + u(x-5)$$

$$\mathcal{L}\{y''\} + 5\,\mathcal{L}\{y'\} + 6\,\mathcal{L}\{y\} = \mathcal{L}\{x \cdot u(x-1)\} - \mathcal{L}\{x \cdot u(x-5)\} + \mathcal{L}\{u(x-5)\}$$

$$s^2 Y(s) - s\cancel{y(0)}^{\,0} - \cancel{y'(0)}^{\,2} + 5\cdot\big[sY(s) - \cancel{y(0)}^{\,0}\big] + 6Y(s) = e^{-s}\cdot\mathcal{L}\{x+1\} - e^{-5s}\cdot\mathcal{L}\{x+5\} + e^{-5s}$$

$$s^2 Y(s) - 2 + 5sY(s) + 6Y(s) = \frac{e^{-s}}{s^2} + \frac{e^{-s}}{s} - \frac{e^{-5s}}{s^2} - \frac{5e^{-5s}}{s} + e^{-5s}$$

$$(s^2 + 5s + 6)Y(s) = e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right) - e^{-5s}\left(\frac{1}{s^2} + \frac{5}{s} - 1\right) + 2$$

$$Y(s) = e^{-s}\cdot\frac{\frac{1}{s^2} + \frac{1}{s}}{(s+2)(s+3)} - e^{-5s}\cdot\frac{\frac{1}{s^2} + \frac{5}{s} - 1}{(s+2)(s+3)} + \frac{2}{(s+2)(s+3)}$$

$$= \mathcal{L}^{-1}\{Y(s)\}$$

$\frac{1+s}{s^2}$ $\qquad$ $1 + 5s - s^2$

$$y = \mathcal{L}^{-1}\left\{e^{-s}\cdot\frac{1+s}{s^2(s+2)(s+3)}\right\} - \mathcal{L}^{-1}\left\{e^{-5s}\cdot\frac{-s^2+5s+1}{s^2(s+2)(s+3)}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{(s+2)(s+3)}\right\}$$

$$\frac{-s^2+5s+1}{s^2(s+2)(s+3)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2} + \frac{D}{s+3}$$

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$$-s^2+5s+1 = As(s+2)(s+3) + B(s+2)(s+3) + Cs^2(s+3) + Ds^2(s+2)$$

$s=0$: $1 = 6B \rightarrow B = 1/6$

$s=-2$: $-5 = 4C \rightarrow C = -5/4$

$s=-3$: $-23 = -9D \rightarrow D = 23/9$.

$s=1$: $5 = 12A + 2 - 5 + \frac{23}{3} \rightarrow \frac{1}{3} = 12A \rightarrow A = 1/36$

$$\mathcal{L}^{-1}\left(\right) = \frac{1}{36}\cdot 1 + \frac{1}{6}\cdot x - \frac{5}{4}e^{-2x} + \frac{23}{9}e^{-3x}$$

$$\left(\frac{1}{36} + \frac{1}{6}(x-5) - \frac{5}{4}e^{-2(x-5)} + \frac{23}{9}e^{-3(x-5)}\right)\cdot u(x-5)$$