Basic Definitions and Terminology.

Algebraic Equations:

$$x^2 - 4 = 5x$$

Differential Equations:

$$y' = y$$

A solution : $y = e^x$

A D.E. is an equation that involve a function and its derivatives.

$$y'' = x$$

The order of a D.E. is the order of the highest derivative in the equation.

A linear D.E. is an equation of the form:

$$a_0(x)y + a_1(x)y' + a_2(x)y'' + \cdots + a_n(x)\underbrace{y^{(n)}}_{n^{\text{th}}\text{ derivative}} = g(x)$$

Solutions to a D.E.

$y' = y$. $\boxed{y = 7e^x}$ is a solution? Yes. Explicit.

$\boxed{y' = 7e^x}$

$7e^x = 7e^x$

E.g. 2.

(1) $(1+x^2)\boxed{y'} = x\boxed{y}$

Verify that $y = \boxed{\sqrt{1+x^2}}$ is a solution to this D.E.

$$y' = \frac{1}{\cancel{2}\sqrt{1+x^2}}\cdot(\cancel{2}x) = \boxed{\frac{x}{\sqrt{1+x^2}}}$$

$$\underbrace{(1+x^2)\cdot\frac{x}{\sqrt{1+x^2}}}_{\sqrt{1+x^2}\cdot x} = x\sqrt{1+x^2} \quad \checkmark$$

2. $y'' + y = \tan x$

$y = \underbrace{-(\cos x)}_{f}\, \underbrace{\ln(\sec x + \tan x)}_{g}$

Product rule:

$(fg)' = f'g + fg'$

$y' = \sin x \cdot \ln(\sec x + \tan x) + (-\cos x) \cdot \dfrac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}$

$\boxed{(\ln(u))' = \dfrac{u'}{u}}$

$= \sin x \cdot \ln(\sec x + \tan x) - \cos x \cdot \dfrac{\sec x\,\cancel{(\tan x + \sec x)}}{\cancel{\sec x + \tan x}}$

$= \sin x \cdot \ln(\sec x + \tan x) - \cancel{\cos x} \cdot \dfrac{1}{\cancel{\cos x}}$

$\boxed{y' = \sin x \cdot \ln(\sec x + \tan x) - 1}$

$$y' = \sin x \cdot \boxed{\ln(\sec x + \tan x)} - 1$$

$$y'' = \cos x \cdot \ln(\sec x + \tan x) + \sin x \cdot \sec x$$

$$y'' = \boxed{\cos x \cdot \ln(\sec x + \tan x) + \tan x}$$

$$\boxed{y''} + y = \tan x$$

$$\cancel{\cos x \cdot \ln(\sec x + \tan x)} + \tan x + \cancel{(-\cos x)\ln(\sec x + \tan x)} \overset{?}{=} \tan x$$

E.g. 3 $x^2 + y^2 - 25 = 0$ (y is defined implicitly)

$$y y' + x = 0$$

$x^2 + y^2 - 25 = 0$

$\longrightarrow$ take the derivative w.r.t. x of both sides:

$\longrightarrow$ $2x + 2yy' = 0$

$\longrightarrow$ $yy' + x = 0$.

So, the relation: $x^2 + y^2 - 25 = 0$ is an implicit solution to the given ODE.