Lecture 4 - First order linear equations

A first order linear equation is an equation of the form:

$$\boxed{a_1(x) \cdot \frac{dy}{dx} + a_0(x) \cdot y = g(x)}$$

E.g.1. $\underbrace{x}_{a_1(x)} \frac{dy}{dx} + \underbrace{2}_{a_0(x)} y = \underbrace{x^{-3}}_{g(x)}$

Step 1: Divide by the coeff. of $\frac{dy}{dx}$ to obtain an equation of the form:

$$\boxed{\frac{dy}{dx} + P(x)y = Q(x)}$$

$$x\frac{dy}{dx} + 2y = x^{-3}$$

$\longrightarrow$ Divide by x

$$\longrightarrow \frac{dy}{dx} + \underbrace{\boxed{\frac{2}{x}}}_{P(x)} y = \underbrace{\boxed{x^{-4}}}_{Q(x)}$$

Step 2: Calculate the integrating factor:

$$\boxed{I(x) = e^{\int P(x)\,dx}}$$

In our example:

$$I(x) = e^{\int \frac{2}{x}\,dx} = e^{2\ln x} = \left(\cancel{e}^{\cancel{\ln} x}\right)^2 = x^2.$$

$$I(x) = x^2.$$

Step 3: Multiply both sides of the equation at end of Step 1 by the integrating factor.

$$x^2\left(\frac{dy}{dx}+\frac{2}{x}y\right)=\left(x^{-4}\right)\cdot x^2$$

$$x^2\cdot\frac{dy}{dx}+2xy = x^{-2}$$

Step 4: The left hand side will always be

$$\frac{d}{dx}\left[I(x)\cdot y\right]$$

In our case:

$$\frac{d}{dx}\left[x^2 \cdot y\right] = x^{-2}$$

Step 5: Integrate both sides w.r.t. x

$$\cancel{\int \frac{d}{dx}}\left[x^2 \cdot y\right] dx = \int x^{-2}\, dx$$

$$x^2 y = \frac{x^{-1}}{-1} + C = -\frac{1}{x} + C$$

$$x^2 y = -\frac{1}{x} + C \longrightarrow \boxed{y = -\frac{1}{x^3} + \frac{C}{x^2}}$$

Summary of Solution Technique for a 1^{st} order linear ODE:

(1) Get the equation to the form:

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

(2) Find the integrating factor:

$$I(x) = e^{\int P(x)\,dx}$$

(3) Multiply both sides by $I(x)$

(4) LHS $= \frac{d}{dx}\left[I(x)\, y \right]$ (by product rule)

(5) Integrate both sides to get y.

E.g. 2 $\quad y' + (\tan x) y = \cos^2 x \quad ; \quad y(0) = -1$

Step 1: $\quad \frac{dy}{dx} + P(x) y = Q(x)$

$$P(x) = \tan x \quad ; \quad Q(x) = \cos^2 x$$

Step 2: Integrating factor : $I(x) = e^{\int P(x)\, dx}$