$$I(x) = e^{\int \tan x \, dx}$$

$$= e^{\int \boxed{\overbrace{\frac{\sin x}{\cos x} \, dx}^{-du}}} \qquad u = \cos x \ ; \ du = -\sin x \, dx$$

$$= e^{\int -\frac{du}{u}} = e^{-\ln(\cos x)} = \left(\cancel{e}^{\cancel{\ln}(\cos x)}\right)^{-1} = (\cos x)^{-1}$$

$\longrightarrow$ So, $I(x) = \sec x$

Step 3: Multiply both sides by $I(x)$

$$\underbrace{(\sec x)\, y' + (\sec x)(\tan x)\, y}_{\text{product rule}} = \sec x \cos^2 x$$

$$\frac{d}{dx}\left[(\sec x)\, y\right] = \cos x$$

Step 5: Integrate both sides

$$\int \cancel{\frac{d}{dx}} \left[(\sec x)\, y\right] dx = \int \cos x \, dx$$

$$(\sec x)\, y = \sin x + C$$

$$y = \frac{\sin x}{\sec x} + \frac{C}{\sec x}$$

$$y = \sin x \cos x + C \cos x$$

Initial condition: $y(0) = -1$

$$-1 = \cancel{\sin(0)\cos(0)}^{\,0} + C\cos(0)$$

$$-1 = C$$

Solution: $\boxed{y(x) = \sin x \cos x - \cos x.}$

<u>E.g. 3.</u> (Discontinuous Forcing Term)

$$\frac{dy}{dx} + y = Q(x) \; ; \; y(0) = 1$$

where $Q(x) = \begin{cases} 1 & \text{when } 0 \leq x \leq 1 \\ -1 & \text{when } x > 1 \end{cases}$

1^{st}: Solve the equation on the interval $0 \leq x \leq 1$

$$\frac{dy}{dx} + y = 1$$

→ Solve by separation of variables:

$$\frac{dy}{dx} = 1 - y$$

$$\int \frac{dy}{1-y} = \int dx \longrightarrow -\ln(1-y) = x + C$$

$$\longrightarrow e^{\ln(1-y)} = e^{-x-C}$$

$$\longrightarrow 1 - y = Ce^{-x} \longrightarrow y = 1 - Ce^{-x}$$

Use initial condition $y(0) = 1$ to solve for C

$$1 = 1 - C \longrightarrow C = 0 \longrightarrow \text{Solution: } y = 1$$

(Sol on $[0, 1]$)

$2^{\underline{nd}}$ Solve the equation when $x > 1$:

$$\frac{dy}{dx} + y = -1$$

$$\frac{dy}{dx} = -1 - y \longrightarrow \frac{dy}{-1-y} = dx$$

$$\longrightarrow \int \frac{dy}{1+y} = -\int dx \longrightarrow e^{\ln(1+y)} = e^{-x+C}$$

$$\longrightarrow 1 + y = Ce^{-x} \longrightarrow y = Ce^{-x} - 1.$$

3^{rd} Glue 2 solutions together to obtain a continuous function y.

Sol 1: $y = 1$ when $0 \leq x \leq 1$

Sol 2: $y = Ce^{-x} - 1$ when $x > 1$

$$y = \begin{cases} 1 & \text{when } 0 \leq x \leq 1 \\ Ce^{-x} - 1 & \text{when } x > 1 \end{cases}$$

Find C such that the piecewise function y is continuous at $x = 1$.

Plug $x = 1$ into both pieces and set them equal:

$$1 = Ce^{-1} - 1 \rightarrow \frac{C}{e} = 2 \rightarrow C = 2e.$$

Sol:

$$\boxed{y = \begin{cases} 1 & \text{when } 0 \leq x \leq 1 \\ (2e)e^{-x} - 1 & \text{when } x > 1 \end{cases}}$$