Lecture 5 - Exact Equation

<u>Goal</u>: Solve an exact equation:

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

What does it mean for the above equation to be an exact equation?

* Partial derivative and total derivative:

$F(x,y)$: function of 2 variables x and y

<u>E.g.</u> $F(x,y) = 3x^2y + 5xy + y^3 + 5$

The total derivative of F is defined to be:

$$dF = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy$$

partial derivative of F w.r.t. x

derivative of F w.r.t. x treating y as a constant

$$\underbrace{6xy + 5y}_{\frac{\partial F}{\partial x}}$$

partial derivative of F w.r.t. y

derivative of F w.r.t. y treating x as a constant

$$\underbrace{3x^2 + 5x + 3y^2}_{\frac{\partial F}{\partial y}}$$

So, $\underbrace{dF}_{\text{total derivative of } F} = \underbrace{(6xy + 5y)dx + (3x^2 + 5x + 3y^2)\,dy}_{\text{this is called a differential expression}}$

Take this diff. expression, set it equal to 0

we get:

$$\underbrace{(6xy + 5y)dx + (3x^2 + 5x + 3y^2)\,dy = 0}_{\text{differential equation}}$$

So, an exact equation is an equation of the form

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

where the left hand side is equal to the total differential of a function $F(x,y)$

The solution to an exact equation will be:

$$F(x,y) = C, \quad C \text{ is any constant.}$$

E.g. 1 (Cont.)

$$F(x,y) = x^2 y - \tan x + y^2 .$$

* Find dF.

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy$$

$$= (2xy - \sec^2 x) dx + (x^2 + 2y) dy$$

* $(2xy - \sec^2 x) dx + (x^2 + 2y) dy = 0$

is an exact equation whose solution is the 1-parameter family: $x^2 y - \tan x + y^2 = C$