Given an equation of the form

$$\boxed{M(x,y)}\,dx + \boxed{N(x,y)}\,dy = 0$$

How do we know if it is exact?

Answer: The equation is exact if

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

We know that $\underbrace{\boxed{(2xy - \sec^2 x)}}_{M}\,dx + \underbrace{\boxed{(x^2 + 2y)}}_{N}\,dy = 0$ is exact. Let see if this test works here?

$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}.$$

Why does it work?

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is exact iff the LHS $= dF$ for some $F(x,y)$

iff $M\,dx + N\,dy = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy$

iff $M = \frac{\partial F}{\partial x}$ and $N = \frac{\partial F}{\partial y}$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) \rightarrow \text{partial of } F \text{ w.r.t. } x \text{ and then partial of result w.r.t } y$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right) \rightarrow \text{partial of } F \text{ w.r.t. } y \text{ and then partial of result w.r.t. } x$$

From cal 3: mixed partials are always equal.

Method of Solution for an exact equation:

E.g. 2 $\underbrace{\boxed{(3x^2y + 8xy^2)}}_{M}dx + \underbrace{\boxed{(x^3 + 8x^2y + 12y^2)}}_{N}dy = 0$

Step 1: Is it exact?

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= 3x^2 + 16xy \\ \frac{\partial N}{\partial x} &= 3x^2 + 16xy \end{aligned}\right\} \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Hence the equation is exact.

Step 2: Since this is exact, we know that

$$M = \frac{\partial F}{\partial x}$$ where F is the function we try to find

So, $F(x,y) = \int M\,dx + \boxed{g(y)}$ constant of integration, a function in y

Here: $M = 3x^2y + 8xy^2$.

So, $F(x,y) = \int (3x^2y + 8xy^2)\,dx + g(y)$

$$F(x,y) = x^3y + 4x^2y^2 + g(y)$$

Step 3: We will find $g(y)$

We know that $N = \frac{\partial F}{\partial y}$

We will take the partial of F w.r.t. y, set it equal to N and then solve for g.

$$F(x,y) = x^3y + 4x^2y^2 + g(y)$$

$$\frac{\partial F}{\partial y} = x^3 + 8x^2y + g'(y)$$

Set this equal to N:

$$\underbrace{\cancel{x^3} + \cancel{8x^2y} + g'(y)}_{\frac{\partial F}{\partial y}} = \underbrace{\cancel{x^3} + \cancel{8x^2y} + 12y^2}_{N}$$