Initial condition: $y(0) = e$

$$e^2 \sin 0 - \cancel{0^3 \cdot e} - 0^2 + \underbrace{e\ln(e) - e}_{0} = C$$

$C = 0$

Solution: $\boxed{y^2 \sin x - x^3 y - x^2 + y \ln y - y = 0}$

Make some non exact equations exact

E.g.4. ① $\underbrace{\boxed{(x^2+y^2+x)}}_{M}dx + \underbrace{\boxed{xy}}_{N}dy = 0$ non exact

$$\frac{\partial M}{\partial y} = 2y \;;\; \frac{\partial N}{\partial x} = y \to \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Note:

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y - y}{xy} = \frac{\cancel{y}}{x\cancel{y}} = \frac{1}{x}$$

Function of x alone

$$\longrightarrow I(x) = e^{\int \frac{1}{x}\,dx} = e^{\ln x} = x$$

$\longrightarrow$ Multiply both sides by this integrating factor:

$$\underbrace{(x^3 + xy^2 + x^2)}_{\text{New } M}\,dx + \underbrace{x^2y}_{\text{New } N}\,dy = 0$$

$$\frac{\partial M}{\partial y} = 2xy \qquad \frac{\partial N}{\partial x} = 2xy \longrightarrow \text{Exact}$$

→ Use the method in previous example to solve.

(2) $\boxed{\cos x}\, dx + \boxed{\left(1 + \frac{2}{y}\right)\sin x}\, dy = 0$

$M \qquad\qquad N$

$$\frac{\partial M}{\partial y} = 0 \quad ; \quad \frac{\partial N}{\partial x} = \left(1 + \frac{2}{y}\right)\cos x$$

$$\longrightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \rightarrow \text{not exact.}$$

But: $$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{\left(1 + \frac{2}{y}\right)\cancel{\cos x}}{\cancel{\cos x}} = \boxed{1 + \frac{2}{y}}$$

function of y alone

So, we can find an integrating factor to make it exact:

$$I(y) = e^{\int \left(1 + \frac{2}{y}\right) dy}$$

$$= e^{y + 2\ln y} = e^y \cdot \left(e^{\ln y}\right)^2 = \boxed{y^2 e^y}$$

→ Multiply both sides of original by this:

$$y^2 e^y \cos x \, dx + y^2 e^y \left(1 + \frac{2}{y}\right) \sin x \, dy = 0$$

new M new N

$$\frac{\partial M}{\partial y} = \left(2y e^y + y^2 e^y\right) \cos x \quad \Big| \quad \frac{\partial N}{\partial x} = \left(y^2 e^y + 2y e^y\right) \cos x$$