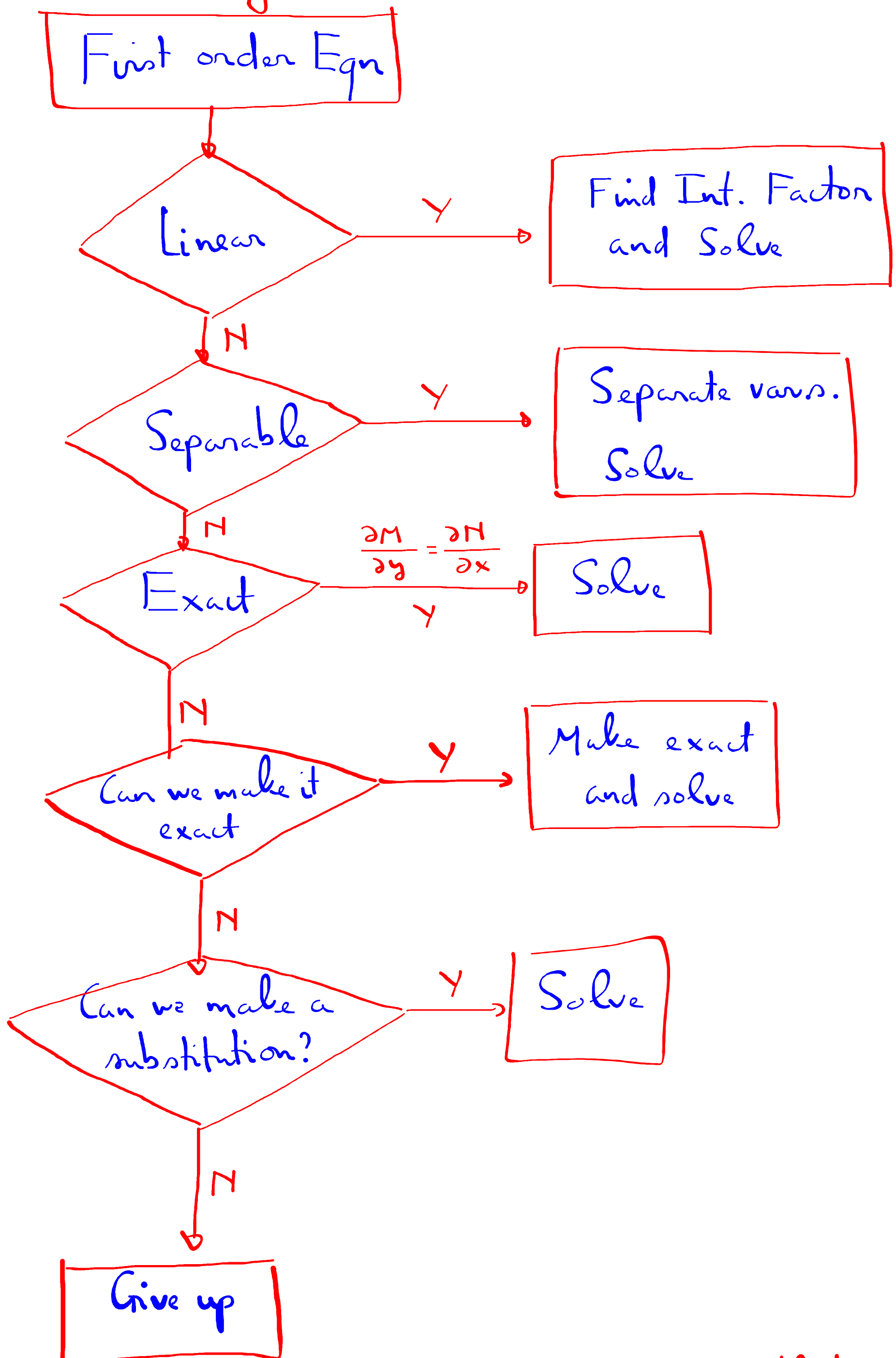
Lecture 6 - Solution by Substitutions



(means: numerical method, geometric method (direction fields))

First order Homogeneous Equation

The equation: $\frac{dy}{dx} = f(x,y)$

is called homogeneous if $f(x,y)$ can be expressed as a function of $\frac{y}{x}$ alone.

E.g. 1: $(y^2 + yx)\,dx + x^2\,dy = 0$

Non linear, Non separable, $\frac{\partial M}{\partial y} = 2y + x$, $\frac{\partial N}{\partial x} = 2x$ $\Big\}$ $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ non exact

$$x^2\,dy = -(y^2 + yx)\,dx$$

$$\frac{dy}{dx} = -\frac{y^2 + yx}{x^2} = -\left(\frac{y^2}{x^2} + \frac{y\cancel{x}}{x^{\cancel{2}}}\right)$$

$$\frac{dy}{dx} = \underbrace{-\left(\left(\frac{y}{x}\right)^2 + \frac{y}{x}\right)}_{\text{this is a function of } \frac{y}{x} \text{ alone}}$$

$\longrightarrow$ this is a homogeneous 1^{st} order equation

Method of solution:

1. let $u = \frac{y}{x}$

(2) $y = ux$

$\longrightarrow$ Take derivative w.r.t. x of both sides:

$$\frac{dy}{dx} = \frac{d}{dx}\left[u \cdot x\right] = x\frac{du}{dx} + u$$

(3) Plug everything back to the original equation:

$$x\frac{du}{dx} + u = -\left(u^2 + u\right)$$

(4) Solve for the function u: this is a separable equation.

$$x\frac{du}{dx} = -u^2 - 2u$$

$$\frac{du}{-u^2-2u} = \frac{dx}{x} \rightarrow \underbrace{\int \frac{du}{u(u+2)}}_{\text{partial fractions decomposition}} = \underbrace{-\int \frac{dx}{x}}_{-\ln(x)+C}$$

$$\frac{1}{u(u+2)} = \frac{A}{u} + \frac{B}{u+2}$$

$$1 = A(u+2) + Bu$$

$u=0$: $A = 1/2$. $u=-2$: $B = -1/2$

$$\int \frac{1}{u(u+2)} = \int \frac{1/2}{u} - \int \frac{1/2}{u+2} = \frac{1}{2}\ln u - \frac{1}{2}\ln(u+2)$$