$$\frac{1}{2}\ln u - \frac{1}{2}\ln(u+2) = -\ln(x) + C$$

$$\longrightarrow \boxed{\frac{1}{2}\ln\left(\frac{y}{x}\right) - \frac{1}{2}\ln\left(\frac{y}{x}+2\right) = -\ln(x) + C}$$

E.g. 2.

$$\frac{dy}{dx} = \frac{(x+3y)/x}{(3x+y)/x} = \frac{\frac{x}{x} + 3\frac{y}{x}}{\frac{3x}{x} + \frac{y}{x}}$$

$$\frac{dy}{dx} = \boxed{\frac{1 + 3\cdot\frac{y}{x}}{3 + \frac{y}{x}}} \longrightarrow \text{Function of } \frac{y}{x} \text{ alone}$$

$\rightarrow$ Homogeneous

Let $u = \frac{y}{x}$. $\quad \frac{dy}{dx} = u + x\frac{du}{dx}$

$$u + x\frac{du}{dx} = \frac{1+3u}{3+u}$$

$$x\frac{du}{dx} = \frac{1+3u}{3+u} - \frac{u}{1}\cdot\frac{(3+u)}{(3+u)}$$

$$x\frac{du}{dx} = \frac{1+\cancel{3u}-\cancel{3u}-u^2}{3+u}$$

$$x\frac{du}{dx} = \frac{1-u^2}{3+u} \longrightarrow \text{Separable.}$$

$$\boxed{\int \frac{3+u}{1-u^2}\,du} = \underbrace{\int \frac{dx}{x}}_{\ln(x)+C}$$

Partial fraction decomp

$$\frac{3+u}{1-u^2} = \frac{3+u}{(1-u)(1+u)} = \frac{A}{1-u} + \frac{B}{1+u}$$

$$3+u = A(1+u) + B(1-u)$$

$u = -1$: $2 = 2B \rightarrow B = 1$

$u = 1$: $4 = 2A \rightarrow A = 2$

$$\int \frac{3+u}{1-u^2} = \int \frac{2}{1-u} + \int \frac{1}{1+u}$$

$$= -2\ln(1-u) + \ln(1+u)$$

Solution: $\boxed{-2\ln\left(1-\frac{y}{x}\right) + \ln\left(1+\frac{y}{x}\right) = \ln x + C}$

Bernoulli's equation:

Equation of the form:

$$\frac{dy}{dx} + P(x)\, y = Q(x) \cdot y^n .$$

Note: if $n \geq 2$; this is a non linear equation.

E.g. 3: $$\frac{dy}{dx} - y = e^{2x} \cdot y^3$$

This is Bernoulli : $P(x) = -1$; $Q(x) = e^{2x}$;
$n = 3$

Step 1: Divide both sides by y^n (multiply by y^{-n})

In this case: y^3 (Multiply by y^{-3})

$$y^{-3}\frac{dy}{dx} - y^{-3}y = e^{2x}\cdot y^3 \cdot y^{-3}$$

$$y^{-3}\frac{dy}{dx} - y^{-2} = e^{2x}.$$

(2) Make a substitution:

$$\boxed{u = y^{-2}} \longrightarrow \frac{du}{dx} = \frac{d}{dx}\left(y^{-2}\right) = -2\boxed{y^{-3}\frac{dy}{dx}}$$