③ Suppose: $y(0) = 0; \quad y'(0) = 1; \quad y''(0) = 3$

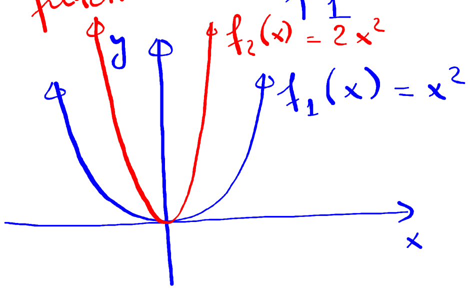
$0 = c_1$

$1 = c_2$

$\cancel{3 = 2}$

Linear Dependence / Linear Independence:

2 functions: $f_1(x)$; $f_2(x)$



linearly dependent

f_1, f_2 are L.D. if 1 function is a constant multiple of the other

f_1, f_2, f_3.

I can find constants c_1, c_2, c_3 that are not all 0 s.t.

$$c_1 f_1 + c_2 f_2 + c_3 f_3 = 0$$

then we say f_1, f_2, f_3 are Dependent.

E.g. $c_1 = 2$; $c_3 = -3$; $c_4 = 10$

$$2 f_1 - 3 f_2 + 10 f_3 = 0$$

E.g. 3.

$f_1 = x^2$; $f_2 = x^2 - 1$, $f_3 = 5$; $I = (-\infty, \infty)$

Q: Dependent or Independent ?

$c_1 = -5$; $c_2 = 5$; $c_3 = 1$

$$c_1 f_1 + c_2 f_2 + c_3 f_3 = -5x^2 + 5(x^2 - 1) + 1 \cdot 5$$

$$= 0$$

$\longrightarrow$ there 3 functions are dependent.

(2) $\{\sin x, \cos x, \tan x\}$; $I = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\boxed{c_1 \sin x + c_2 \cos x + c_3 \tan x = 0} ; \quad I = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

This has to be true for all x in I.

$$x = 0 : \quad c_1 \cdot \underbrace{\sin(0)}_{0} + c_2 \underbrace{\cos(0)}_{1} + c_3 \underbrace{\tan(0)}_{0} = 0$$

$$\boxed{c_2 = 0}$$

$$x = \frac{\pi}{4} : \quad c_1 \sin\left(\frac{\pi}{4}\right) + c_3 \tan\left(\frac{\pi}{4}\right) = 0$$

$$c_1 \cdot \frac{\sqrt{2}}{2} + c_3 = 0 \longrightarrow c_1 \frac{3\sqrt{2}}{2} + \cancel{3 c_3} = 0$$

$$x = \frac{\pi}{3} : \quad c_1 \sin\left(\frac{\pi}{3}\right) + c_3 \tan\left(\frac{\pi}{3}\right) = 0$$

$$c_1 \cdot \frac{\sqrt{3}}{2} + c_3 \sqrt{3} = 0 \longrightarrow c_1 \frac{3}{2} + \cancel{3 c_3} = 0$$

$\Big\}\ \ominus$

$$c_1 \left(\frac{3\sqrt{2}}{2} - \frac{3}{2}\right) = 0$$

$$\longrightarrow \boxed{\begin{array}{l} c_1 = 0 \\ c_3 = 0 \end{array}}$$

E.g. 4.

$4y'' - 4y' + y = 0$

① Verify that $y = e^{x/2}$ and $y = xe^{x/2}$ are solutions:

$y = e^{x/2}$: $y' = \frac{1}{2}e^{x/2}$; $y'' = \frac{1}{4}e^{x/2}$

$4\left(\frac{1}{4}e^{x/2}\right) - 4\left(\frac{1}{2}e^{x/2}\right) + e^{x/2} \overset{?}{=} 0$

$\underbrace{e^{x/2} - 2e^{x/2} + e^{x/2}}_{0} = 0$

0. So, $y = e^{x/2}$ is a solution.

$y = x \cdot e^{x/2}$: $y' = e^{x/2} + \frac{1}{2}xe^{x/2}$

$y'' = \frac{1}{2}e^{x/2} + \frac{1}{2}\left(e^{x/2} + \frac{1}{2}xe^{x/2}\right)$

$= e^{x/2} + \frac{1}{4}xe^{x/2}$

$4\left(e^{x/2} + \frac{1}{4}xe^{x/2}\right) - 4\left(e^{x/2} + \frac{1}{2}xe^{x/2}\right) + xe^{x/2} \overset{?}{=} 0$

$\cancel{4e^{x/2}} + \cancel{xe^{x/2}} - \cancel{4e^{x/2}} - \cancel{2xe^{x/2}} + \cancel{xe^{x/2}} = 0$ ✓

So, $y = e^{x/2}$ and $y = xe^{x/2}$ are solutions to the equation.

(2) Are they independent? Yes, b/c one is not a constant multiple of the other.

(3) $\boxed{y = c_1 e^{x/2} + c_2 x e^{x/2}}$ → the general solution to the equation.

$\{e^{x/2}, xe^{x/2}\}$ → Fundamental set of solutions

E.g. 5 $\boxed{4y'' - 4y' + y} = 25 \sin x$ → Nonhomogeneous.

Associate Homogeneous equation: $4y'' - 4y' + y = 0$

→ Fund. set of solutions to homo. $\{e^{x/2}, xe^{x/2}\}$