Lecture 8 : Reduction of order

E.g. 1. $y'' - 25y = 0$

Given: $y_1 = e^{5x}$ is a solution.

Use the reduction of order method to find a second solution y_2 that is independent of y_1.

1 Assume : $y_2 = u \cdot y_1 = u \cdot e^{5x}$

$\rightarrow$ Goal: solve for the function $u(x)$

$y = u \cdot e^{5x}$

(2) Substitute $y = u \cdot e^{5x}$ into the equation:

$$y'' - 25y = 0$$

$$y' = u' \cdot e^{5x} + u \cdot 5e^{5x} = e^{5x}u' + 5ue^{5x}$$

$$y'' = u'' \cdot e^{5x} + \underbrace{u' \cdot 5e^{5x} + 5u'e^{5x}} + 25ue^{5x}$$

$$\boxed{y'' = e^{5x} \cdot u'' + 10e^{5x}u' + 25e^{5x} \cdot u}$$

$$\underbrace{e^{5x} \cdot u'' + 10e^{5x}u' + \cancel{25e^{5x}} \cdot u}_{y''} \underbrace{- \cancel{25e^{5x}} \cdot u}_{-25y} = 0$$

$$\underbrace{e^{5x}}_{>0}\left[u'' + 10u'\right] = 0$$

$$\longrightarrow u'' + 10u' = 0$$

(3) let $w = u'$

$$w' + 10w = 0$$

Integrating factor: $I = e^{\int 10\,dx} = e^{10x}$

Multiply both sides by I:

$$\underbrace{e^{10x} w' + 10 e^{10x} w} = 0$$

$$\int \frac{d}{dx}\left[e^{10x} w\right] = \int 0$$

$$e^{10x} w = C \longrightarrow w = Ce^{-10x}$$

Since $w = u'$, $\quad u' = Ce^{-10x}$

$$u = \int C e^{-10x}\,dx = C\int e^{-10x}\,dx$$
$$= C\left(-\frac{1}{10}e^{-10x}\right)$$

Conclusion: $y_2 = -\frac{C}{10}e^{-10x}\cdot e^{5x} = -\frac{C}{10}e^{-5x}$

General solution to the equation:

$$y = c_1 y_1 + c_2 y_2$$

constant

$$= c_1 e^{5x} + c_2\cdot\left(-\frac{C}{10}e^{-5x}\right)$$

$$\boxed{y = c_1 e^{5x} + c_2 e^{-5x}}$$

E.g. 2: $x^2y'' - xy' + y = 0$

Given: $y_1 = x$ is a solution.

Find a second ind. solution.

Reduction of order: Assume: $y_2 = u \cdot y_1 = u \cdot x$

$y = ux$.

$y' = u'x + u$. $\quad y'' = u''x + u' + u'$

$= u''x + 2u'$

$$x^2\underbrace{(u''x + 2u')}_{y''} - x \cdot \underbrace{(u'x + u)}_{y'} + \underbrace{ux}_{y} = 0$$

$$x^3u'' + 2x^2u' - x^2u' - \cancel{ux} + \cancel{ux} = 0$$

$$x^3u'' + x^2u' = 0$$

$$x^2(xu'' + u') = 0$$

Since $x > 0$, divide by x^2: $\quad xu'' + u' = 0$

Let $w = u'$. This becomes: $xw' + w = 0$

$$x w' + w = 0$$

$$w' + \frac{1}{x} w = 0$$

$$I = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$$

$$x w' + w = 0$$

$$\int \frac{d}{dx}[x w] = \int 0 \longrightarrow x w = C$$

$$w = \frac{C}{x}$$

$$u' = \frac{C}{x} \longrightarrow u = \int \frac{C}{x} dx = C \ln x$$