$$y_2 = u\, y_1 = C x \ln x$$

General solution:

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 x + c_2 \cdot C x \ln x$$

$$\boxed{y = c_1 x + c_2 x \ln x}$$

E.g. 3. $y'' + 9y = 0$

Given: $y_1 = \sin(3x)$

Use formula to find y_2.

$$y_2 = y_1 \cdot \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx$$

$$y_2 = \sin(3x) \cdot \int \frac{e^{-\int 0\,dx}}{\sin^2(3x)}\,dx$$

$$= \sin(3x) \cdot \int \frac{C}{\sin^2(3x)}\,dx$$

$$= C \sin(3x) \cdot \int \csc^2(3x)\,dx$$

$$= C \sin(3x) \left(-\frac{\cot(3x)}{3} \right)$$

$$= C \sin(3x) \cdot \cot(3x) = C \cos(3x)$$

General solution:

$$\boxed{y = c_1 \sin(3x) + c_2 \cos(3x)}$$

Derivation of formula:

$$y'' + P(x)\, y' + Q(x)\, y = 0$$

Given a solution y_1.

Find y_2 by reduction of order.

$$y_2 = u \cdot y_1 \rightarrow y = u \cdot y_1$$

$$y' = u' \cdot y_1 + u \cdot y_1'$$

$$y'' = u'' \cdot y_1 + u' y_1' + u' y_1' + u \cdot y_1''$$

$$= u'' \cdot y_1 + 2u' \cdot y_1' + u \cdot y_1''$$

$$\underbrace{\boxed{u'' \cdot y_1 + 2u' \cdot y_1' + u \cdot y_1''}}_{y''} + P \cdot \Big(\overbrace{\boxed{u' y_1 + u y_1'}}^{y'}\Big) + Q \underbrace{\boxed{u y_1}}_{y} = 0$$

$$u'' \cdot y_1 + 2u' y_1' + u y_1'' + P u' y_1 + P u y_1' + Q u y_1 = 0$$

$$u'' \cdot y_1 + 2u' y_1' + P u' y_1 + \cancel{u \cdot \underbrace{\boxed{\left(y_1'' + P y_1' + Q y_1 \right)}}_{0 \text{ (b/c } y_1 \text{ is a solution)}}} = 0$$

$$y_1 u'' + (2y_1' + P y_1) u' = 0$$

let $w = u'$

$$y_1 w' + (2y_1' + P y_1) w = 0$$

$$w' + \left(\frac{2y_1' + P y_1}{y_1} \right) w = 0$$

$$I = e^{\int \frac{2y_1' + P y_1}{y_1} dx} = e^{2 \int \frac{y_1'}{y_1} dx + \int P dx}$$

$$= e^{2 \ln(y_1) + \int P dx} = y_1^2 \cdot e^{\int P dx}$$

$$\int \frac{d}{dx}\left[w \cdot y_1^2 e^{\int P\,dx}\right] = \int 0$$

$$w \cdot y_1^2 e^{\int P\,dx} = C$$

$$u' = w = \frac{C}{y_1^2 e^{\int P\,dx}} = \frac{C}{y_1^2} e^{-\int P\,dx}$$

$$u = C \int \frac{e^{-\int P\,dx}}{y_1^2}\,dx$$

$$y_2 = u y_1 = C y_1 \cdot \int \frac{e^{-\int P\,dx}}{y_1^2}\,dx$$