Lecture 9: Homogeneous linear Equations with constant coefficients.

Equation of order 2:

$$ay'' + by' + c = 0$$

a, b, c are constants and $a \neq 0$.

Assumption: Solution is a function of the form

$$y = e^{mx}$$

Substitute y, y', y'' into equation:

$$y' = me^{mx} \quad ; \quad y'' = m^2 e^{mx}$$

$$\longrightarrow am^2 e^{mx} + bme^{mx} + ce^{mx} = 0$$

$$\boxed{e^{mx}}\left(am^2 + bm + c\right) = 0$$

> 0

So, $am^2 + bm + c = 0$.

If $y = e^{mx}$ is a solution to the equation, then m must satisfy the algebraic equation:

$$\boxed{am^2 + bm + c = 0} \longrightarrow \text{characteristic equation of the D.E.}$$

An algebraic equation in vari. m

Quadratic equation

$$m_1 = \frac{-b+\sqrt{b^2-4ac}}{2a} \; ; \quad m_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

3 cases:

Case 1: $b^2 - 4ac > 0$: 2 solutions: real, distinct to the char. eqn. m_1, m_2.

Solution to the D.E.: $y = e^{m_1 x}$; $y = e^{m_2 x}$

These are L.I.

The general solution to the D.E.

$$\boxed{y = c_1 e^{m_1 x} + c_2 e^{m_2 x}}$$

Case 2: $b^2 - 4ac = 0$. 1 real solution to the

Char. eqn: $m_1 = m_2 = -\frac{b}{2a}$.

$\rightarrow$ 1 solution of the D.E. $y = e^{m_1 x}$

Apply the Reduction of Order method to find a second L.I. solution.

$\rightarrow$ result: second solution: $y = x e^{m_1 x}$

The general solution to the DE:

$$\boxed{y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}}$$

Case 3: $b^2 - 4ac < 0 \rightarrow$ 2 complex (non-real) solutions:

The complex (non real) solutions appear as a conjugate pair : $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$

2 L.I. solution to DE :

$$y = e^{(\alpha + i\beta)x} \quad ; \quad y = e^{(\alpha - i\beta)x}$$

General solution to DE :

$$y = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x}$$

Apply Euler's formula: $\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$

To simplify the above expression, we obtain the general formula for the D.E. :

$$y = c_1 e^{\alpha x}\sin(\beta x) + c_2 e^{\alpha x}\cos(\beta x)$$

$$= e^{\alpha x}\left(c_1 \sin(\beta x) + c_2 \cos(\beta x)\right)$$

<u>E.g.</u> $m_1 = 2 - 3i$; $m_2 = 2 + 3i$

$\longrightarrow$ general solution to DE:

$$y = e^{2x}\left(c_1 \sin(3x) + c_2 \cos(3x)\right)$$