Definitions and Terminology

Recommended reading from Zill's DEs with BVP-7e: Section 1.1 (pg 2-9): Examples 1 through 4. Section 1.2 (pg 13-17): Examples 1 through 3.

Ordinary Differential Equations - Definitions

Let $y = f(x)$ be a function of x. An **ordinary differential equation (ODE)** is an equation involving x, the function $y = f(x)$ and one or more of its derivatives.
The **order** of an ODE is the order of the highest derivative in the equation.
A **linear ODE** is an equation of the form

$$a_0(x)y + a_1(x)y' + a_2(x)y'' + \cdots + a_n(x)y^{(n)}(x) = g(x).$$

In a linear ODE, the power of each term involving y and its derivatives $y', y'', \ldots$ is at most 1 and the coefficients $a_0, a_1, \ldots, a_n$ either are constant or depend only on x.
A **nonlinear ODE** is an ODE that is not linear. Nonlinear functions of the dependent variable y or its derivative such as $y^2, \cos(y), e^y, \ln(y), \arctan(y'), (y')^3$ cannot appear in a linear ODE.

Example 1: Definitions and terminology

Determine the order of the ODE and determine whether it is linear or nonlinear. If it is nonlinear, explain why.

1. $\dfrac{dy}{dx} = 5y$
2. $y' + 2xy = x^3$
3. $m\dfrac{d^2y}{dx^2} + c\dfrac{dy}{dx} + ky = F(x)$; m, c, k are constants.
4. $(\sin x)y''' - (\cos x)y' = 2$
5. $y'' - \left[1 - (y')^2\right] y' + y = 0$
6. $y'' = e^y$

Solution

Write the solution here

Solution of an ODE - Explicit vs. Implicit

The function $y = f(x)$ defined on an interval I is an **explicit solution** of an ODE if replacing y by $f(x)$, y' by $f'(x), \ldots, y^{(n)}$ by $f^{(n)}(x)$ reduces the equation to an identity.
A relation $G(x, y) = 0$ is an **implicit solution** of an ODE on an interval $I = (a, b)$ if

1. The relation defines y as an implicit function of x on I, i.e., there exists a function $y = \phi(x)$ that satisfies the relation and
2. $y = \phi(x)$ satisfies the ODE on I.

Example 2: Explicit solution

Verify that the indicated function is an explicit solution of the given ODE.

1. $(1+x^2)y' = xy; \quad y = \sqrt{1+x^2}$
2. $y'' + y = \tan x; \quad y = -(\cos x)\ln(\sec x + \tan x)$

Solution

Write the solution here

Example 3: Implicit solution

Determine whether the relation $G(x, y) = x^2 + y^2 - 25 = 0$ is an implicit solution of the ODE

$$yy' + x = 0 \text{ on the interval } I = (-5, 5).$$

Solution

Write the solution here

Family of solutions - general solution - particular solution - initial conditions

A solution containing a constant c is called a **one-parameter family of solutions**. A solution containing n contants $c_1, c_2, \ldots, c_n$ is called a **n-parameter family of solutions**. The constants $c_1, \ldots, c_n$ are the parameters.
A solution that does not contain parameters (constants) is called a **particular solution**.
If every solution of an ODE can be obtained from an n-parameter family of solutions by appropriate choices of the parameters, then the family is called the **general solution** of the ODE.
An **initial value problem (IVP)** is a problem in which we find a solution $y(x)$ of an ODE so that $y(x)$ also satisfies certain prescribed conditions, i.e., conditions that are imposed on the values of the unknown function $y(x)$ and its derivatives at a specific number x_0. These conditions are called the **initial conditions** of the ODE and for an order nth ODE are of the form

$$y(x_0) = y_0, y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1},$$

where $y_0, y_1, \ldots, y_{n-1}$ are constants.

Example 4: Family of solutions

Verify that the 3-parameter family of functions $y = c_1 x^{-1} + c_2 x + c_3 x \ln x + 4x^2$ is a family of solutions to the ODE

$$x^3 y''' + 2x^2 y'' - xy' + y = 12x^2.$$

Solution

Write the solution here

Example 5: Particular solution

Given that $y = c_1 \cos(x) + c_2 \sin(x)$ is a 2-parameter family of solutions of the ODE $y'' + y = 0$. Find the particular solution of the ODE that satisfies the initial conditions $y(\pi/2) = 0$ and $y'(\pi/2) = 1$.

Solution

Write the solution here

Example 6: Find a differential equation

Find a differential equation whose 2-parameter family of solutions is

$$y = c_1 e^{x} + c_2 e^{-x}.$$

Solution

Write the solution here