Nonhomogeneous Equations: the Method of Undetermined Coefficients

Solving a nonhomogeneous linear equation

Consider a nonhomogeneous linear equation

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1 y' + a_0 y = g(x),$$

where a_n, $a_{n-1}, \ldots, a_0$ are real constants with $a_n \neq 0$. If $g(x) \neq 0$ on an interval I, then the result from Lecture 9 says that to solve this equation on I, we must do two things:

- Find the complementary function y_c which is the general solution to the associated homogeneous equation.
- Find a particular solution y_p of the nonhomogeneous equation.

The general solution to the nonhomogeneous equation will then be

$$y(x) = y_c(x) + y_p(x).$$

We learned how to find y_c in Lecture 9. In this lecture, we will learn the **method of undetermined coefficients** to find y_p. This method applies in the case when $g(x)$ is a function or a sum of functions each of which has a finite number of linearly independent derivatives. In particular, $g(x)$ is a a sum of functions $g(x) = f_1(x) + \cdots + f_l(x)$ where each $f_j(x)$ is one of the forms

$$P(x) = a_n x^n + \cdots + a_1 x + a_0, \text{(a polynomial), or}$$

$$P(x)e^{\alpha x} \text{(a polynomial times an exponential function), or}$$

$$P(x)e^{\alpha x}\sin(\beta x) \text{ or } P(x)e^{\alpha x}\cos(\beta x) \text{(a polynomial times an exponential function times a trig function).}$$

Method of Undetermined Coefficients - Case 1 - No term of $g(x)$ is the same as a term of y_c

Suppose that $g(x) = f_1(x) + \cdots + f_l(x)$ and no term of $g(x)$ is the same as a term of the complementary function y_c. The particular solution y_p has the form

$$y_p = y_{p_1} + \cdots + y_{p_l}$$

where each y_{p_j} will be a linear combination $f_j(x)$ and all its linearly independent derivatives. In particular,

- If $f_j = a_n x^n + \cdots + a_1 x + a_0$ (a polynomial), then the form of y_{p_j} is

  $$y_{p_j} = B_n x^n + \cdots + B_1 x + B_0,$$

  where the coefficients $B_n, \ldots, B_1, B_0$ are to be determined (hence the name *undetermined coefficients*).
- If $f_j = (a_n x^n + \cdots + a_1 x + a_0)e^{\alpha x}$, then the form of y_{p_j} is

  $$y_{p_j} = (B_n x^n + \cdots + B_1 x + B_0)e^{\alpha x},$$

  where the coefficients $B_n, \ldots, B_1, B_0$ are to be determined.
- If $f_j = (a_n x^n + \cdots + a_1 x + a_0)e^{\alpha x}\sin(\beta x)$ or $f_j = (a_n x^n + \cdots + a_1 x + a_0)e^{\alpha x}\cos(\beta x)$, then

  $$y_{p_j} = (B_n x^n + \cdots + B_1 x + B_0)e^{\alpha x}\sin(\beta x) + (E_n x^n + \cdots + E_1 x + E_0)e^{\alpha x}\cos(\beta x),$$

  where the coefficients $B_n, \ldots, B_1, B_0$ and $E_n, \ldots, E_1, E_0$ are to be determined.

Example 1: Form of a particular solution - Case 1

Determine the form of a particular solution for the given nonhomogeneous equation:

$$y'' + 2y' - 3y = g(x)$$

where $g(x)$ equals

1. $7x + x^2 e^{-x}$
2. $2xe^x \sin(x) + 5e^{3x}$
3. $x^2 \cos(\pi x)$

Solution

Write the solution here

Example 2: Solve a nonhomogeneous equation - Case 1

Given the equation $y'' + 4y' + 4y = 4x^2 + 6e^x$.

1. Find the general solution to the associated homogeneous equation.
2. Determine the form of a particular solution.
3. Use the method of undetermined coefficient to find a particular solution.
4. Find the general solution to the nonhomogeneous equation.

Solution

Write the solution here

Example 3: Solve a nonhomogeneous equation - Case 1

Find the general solution to the equation: $y''' + y'' = e^x \cos x$.

Solution

Write the solution here

Case 2 - The normal form of y_p contains terms which duplicate terms in y_c

Suppose that we write the form of the particular solution as

$$y_p = y_{p_1} + \cdots + y_{p_l},$$

where the form of y_{p_j} is based on the function $f_j(x)$. Suppose further that, y_{p_j} contains terms (after ignoring constant coefficients) that duplicates terms in the complementary function y_c, then we need to adjust y_{p_j} by multiplying its normal form by x^k where k is the smallest positive integer that eliminates any duplication.
For example, if $f_j = (a_n x^n + \cdots + a_1 x + a_0)e^{\alpha x}$, then the normal form of y_{p_j} is

$$y_{p_j} = (B_n x^n + \cdots + B_1 x + B_0)e^{\alpha x}.$$

However, if the complementary function y_c has the term $e^{\alpha x}$, we need to adjust y_{p_j} by multiplying it by x. If y_c has the term $xe^{\alpha x}$, we need to multiply y_{p_j} by x^2, and so on. More generally, if y_c has the term $x^k e^{\alpha x}$, we need to multiply y_{p_j} by x^{k+1}. The adjusted y_{p_j} will be

$$y_{p_j} = x^{k+1}(B_n x^n + \cdots + B_1 x + B_0)e^{\alpha x}.$$

Case 2 - Some concrete examples

For example, suppose

$$g(x) = \underbrace{5x^2}_{f_1(x)} + \underbrace{7e^{2x}}_{f_2(x)}$$

and the complementary function is $y_c = c_1e^x + c_2e^{2x}$. The normal form of y_p is

$$y_p = \underbrace{Ax^2 + Bx + C}_{y_{p_1}} + \underbrace{Ee^{2x}}_{y_{p_2}}.$$

Since y_{p_2} contains e^{2x} which duplicates the term e^{2x} in y_c, we need to adjust y_{p_2} by multiplying it by x. Hence, the correct form of the particular solution must be

$$y_p = \underbrace{Ax^2 + Bx + C}_{y_{p_1}} + \underbrace{Exe^{2x}}_{\text{adjusted } y_{p_2}}.$$

As another examples suppose

$$g(x) = \underbrace{3xe^{-2x}}_{f_1(x)} + \underbrace{\sin x}_{f_2(x)}$$

and the complementary function is $y_c = c_1e^{-2x} + c_2xe^{-2x}$. The normal form of y_p is

$$y_p = \underbrace{(Ax + B)e^{-2x}}_{y_{p_1}} + \underbrace{C\sin x + E\cos x}_{y_{p_2}}.$$

Since y_{p_1} contains the terms xe^{-2x} and e^{-2x} both of which duplicate terms in y_c, we need to adjust y_{p_1} by multiplying it by the smallest power of x which eliminates any duplication. Hence, we multiply y_{p_1} by x^2. So, the correct form of y_p must be

$$y_p = \underbrace{x^2(Ax + B)e^{-2x}}_{\text{adjusted } y_{p_1}} + \underbrace{C\sin x + E\cos x}_{y_{p_2}}.$$

Example 4: Form of a particular solution - Case 2

Determine the form of a particular solution for the given nonhomogeneous equation:

1. $y'' - y = 7xe^x + 5e^{-x}$
2. $y'' - 2y' + 2y = 5xe^x\cos x$
3. $y^{(4)} + y''' = 1 - x^2e^{-x}$

Solution

Write the solution here

Example 5: Solve a nonhomogeneous equation - Case 2

Find the general solution to the equation: $y'' + y = 4x \sin x + e^{-x}$.

Solution

Write the solution here

Example 6: Solve a nonhomogeneous equation - Case 2

Find the general solution to the equation: $y''' - 2y'' + y' = x - e^{x}$.

Solution

Write the solution here