# Variation of Parameters

## Method of Variation of Parameters

In the previous lecture, we learned the method of undetermined coefficient to find a particular solution of the nonhomogeneous linear equation

$$a_n \frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1 y' + a_0 y = g(x), a_n \neq 0.$$

For that method to work, $g(x)$ must be a function which has a finite number of linearly independent derivatives. It will not work when $g(x) = \ln x$, $g(x) = \frac{1}{x}$, $g(x) = \tan x$, $g(x) = \arctan x$, and so on. Such functions have an infinite number of linearly independent derivatives. The method of **variation of parameters** can be applied in these situations.
Within the scope of this lecture, we focus on the second order equation with constant coefficients

$$ay'' + by' + cy = g(x), a \neq 0.$$

Suppose that we already found the two linearly independent solutions to the associated homogeneous equation $y_1$ and $y_2$. We assume that a particular solution to the nonhomogeneous take the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x).$$

Substitute $y_p$ into the nonhomogeneous equation, we will obtain a 2-by-2 system of equations in $u_1'$ and $u_2'$. We then solve this system for $u_1'$ and $u_2'$. Finally, we integrate $u_1'$, $u_2'$ to find $u_1$, $u_2$ and use them to obtain $y_p$. Here is a summary of all the formulas that give $u_1$, $u_2$ and $y_p$.
**Method of Variation of Parameters:**

1. Divide both sides of the original equation by the *leading coefficient* $a$ to obtain the equation

$$y'' + By' + Cy = f(x).$$

2. Find the complementary function $y_c(x) = c_1 y_1(x) + c_2 y_2(x)$ of the associated homogeneous equation. This gives us two linearly independent functions $y_1$ and $y_2$.

3. Find the quantities

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2; \quad W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} = -y_2 f(x); \quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix} = y_1 f(x).$$

4. Compute

$$u_1 = \int \frac{W_1}{W} dx = \int \frac{-y_2 f(x)}{y_1 y_2' - y_1' y_2} dx, \quad u_2 = \int \frac{W_2}{W} dx = \int \frac{y_1 f(x)}{y_1 y_2' - y_1' y_2} dx.$$

5. A particular solution is

$$y_p = u_1 y_1 + u_2 y_2.$$

**Note 1:** Variation of parameters can also be used to find a particular solution to a nonhomogeneous second order linear equation with *variable coefficients* if two linearly independent solutions to the associated homogeneous equation are known.
**Note 2:** Variation of parameters can also be used for nonhomogeneous linear equations with order $n$ higher than 2. We will need to solve an $n$-by-$n$ system of equations for $u_1', \ldots, u_n'$. We will not consider this case here.
**Note 3:** Of course we can also use variation of parameters when $g(x)$ has a finite number of linearly independent derivatives. However, in this case, the method of undetermined coefficients is usually easier to use.

**Example 1: Solve a nonhomogeneous equation**

Use variation of parameters to find a particular solution on the interval $(-\pi/2, \pi/2)$ of the given equation. Then find the general solution of the equation:

$$y'' + y = \tan x.$$

**Solution**

The associated homogeneous equation is $y'' + y = 0$. The characteristic equation is $m^2 + 1 = 0$. The roots of the characteristic equation are $m = \pm i$. Hence, the complementary function is $y_c = c_1 \cos x + c_2 \sin x$. This gives us $y_1 = \cos x$ and $y_2 = \sin x$.
From the formulas in Step 3, we calculate

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2(x) + \sin^2(x) = 1; \quad W_1 = \begin{vmatrix} 0 & \sin x \\ \tan x & \cos x \end{vmatrix} = -\sin x \tan x; \quad W_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & \tan x \end{vmatrix} = \cos x \tan x.$$

Then from the formulas in Step 4, we find $u_1$ and $u_2$:

$$\begin{aligned} u_1 &= \int \frac{W_1}{W} dx = -\int \tan x \sin x dx = -\int \frac{\sin^2 x}{\cos x} dx \\ &= -\int \frac{1 - \cos^2 x}{\cos x} dx = \int (\cos x - \sec x)\, dx \\ &= \sin x - \ln|\sec x + \tan x|. \end{aligned}$$

And

$$u_2 = \int \frac{W_2}{W} dx = \int \cos x \tan x dx = \int \sin x dx = -\cos x.$$

Hence, a particular solution is

$$y_p = (\sin x - \ln|\sec x + \tan x|)\cos x - \cos x \sin x = -\cos(x) \ln|\sec x + \tan x|.$$

Thus, the general solution to the equation is

$$y = c_1 \cos x + c_2 \sin x - \cos(x) \ln|\sec x + \tan x|.$$

**Example 2: Solve a nonhomogeneous equation - variable coefficients**

As mentioned, the method of variation of parameters (same formulas) works for a second order equation with variable coefficients. Given that $y_1 = x$ and $y_2 = x \ln x$ are linearly independent solutions of the associated homogeneous equation. Find a particular solution and find the general solution of the nonhomogeneous equation

$$x^2 y'' - x y' + y = x.$$

**Solution**

We first divide both sides of the equation by $x^2$ to obtain

$$y'' - \frac{1}{x} y' + \frac{1}{x^2} y = \frac{1}{x}.$$

We are already given the independent solutions of the associated homogeneous equation $y_1 = x$ and $y_2 = x \ln x$. So, we calculate

$$W = \begin{vmatrix} x & x\ln x \\ 1 & 1 + \ln x \end{vmatrix} = x; \quad W_1 = \begin{vmatrix} 0 & x \ln x \\ \frac{1}{x} & 1 + \ln x \end{vmatrix} = -\ln x; \quad W_2 = \begin{vmatrix} x & 0 \\ 1 & \frac{1}{x} \end{vmatrix} = 1.$$

Then from the formulas in Step 4, we find $u_1$ and $u_2$:

$$u_1 = \int \frac{W_1}{W} dx = -\int \frac{\ln x}{x} dx = -\frac{(\ln x)^2}{2} \quad (u - \text{substitution with } u = \ln x).$$

And

$$u_2 = \int \frac{W_2}{W} dx = \int \frac{1}{x} dx = \ln x.$$

**Solution**

It follows that a particular solution is

$$y_p = -\frac{x}{2}(\ln x)^2 + x(\ln x)^2 = \frac{x}{2}(\ln x)^2.$$

Hence, the general solution to the equation is

$$y = c_1 x + c_2 x \ln x + \frac{x}{2}(\ln x)^2.$$

**Example 3: Solve a nonhomogeneous equation - Variation of parameters and Undetermined Coefficients**

Given the equation $y'' + y = \tan x + e^{3x} - 1$.

1. In Example 1, we have found the general solution to the associated homogeneous equation. We also found a particular solution to the equation $y'' + y = \tan x$. Use the method of undetermined coefficient to find a particular solution to the equation $y'' + y = e^{3x} - 1$.
2. Find the general solution to the given equation.

**Solution**

1. From example 1, the general solution to the associated homogeneous equation is $y = c_1 \cos x + c_2 \sin x$. Moreover, we found a particular solution to the equation $y'' + y = \tan x$, that is, $y_{p_1} = -\cos(x)\ln|\sec x + \tan x|$. Now, we will find a particular solution to the equation $y'' + y = e^{3x} - 1$ by the method of undetermined coefficients.

   The form of $y_{p_2}$ is $y_{p_2} = Ae^{3x} + Bx + C$. So, $y'_{p_2} = 3Ae^{3x} + B$ and $y''_{p_2} = 9Ae^{3x}$. Substitute these into the equation, we have
   $$9Ae^{3x} + Ae^{3x} + Bx + C = 10Ae^{3x} + Bx + C = e^{3x} - 1.$$
   It follows that $10A = 1$, $B = 0$ and $C = -1$. So, $A = 1/10$ and we have $y_{p_2} = \frac{1}{10}e^{3x} - 1$.

2. A particular solution to $y'' + y = \tan x + e^{3x} - 1$ is
   $$y_p = y_{p_1} + y_{p_2} = -\cos(x)\ln|\sec x + \tan x| + \frac{1}{10}e^{3x} - 1.$$
   Hence, the general equation to the equation is
   $$y = c_1 \cos x + c_2 \sin x - \cos(x)\ln|\sec x + \tan x| + \frac{1}{10}e^{3x} - 1.$$