**Review of Power Series** 

A **power series** centered at the point  $x_0$  is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \dots,$$

where x is a variable and the  $a_n$  are real numbers called the coefficients of the series. Throughout this lecture, we assume the center is 0, i.e.,  $x_0 = 0$ . A power series centered at 0 takes the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

The power series representation for some familiar functions are:

$$e^{x} = \sum_{n=0}^{\infty} \frac{x^{n}}{n!} = 1 + x + \frac{x^{2}}{2!} + \frac{x^{3}}{3!} + \dots$$
$$\sin(x) = \sum_{n=0}^{\infty} (-1)^{n} \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^{3}}{3!} + \frac{x^{5}}{5!} + \dots$$
$$\cos(x) = \sum_{n=0}^{\infty} (-1)^{n} \frac{x^{2n}}{(2n)!} = 1 - \frac{x^{2}}{2!} + \frac{x^{4}}{4!} + \dots$$

We say that a power series  $\sum_{n=0}^{\infty} a_n x^n$  converges at a real number, say x = 2 if when we replace x by 2, the infinite

series of real number  $\sum_{n=0}^{\infty} a_n 2^n$  converges. Every power series has an interval of convergence which is the set of all

real numbers x for which the series converges. The interval of convergence of a power series  $\sum_{n=0}^{\infty} a_n x^n$  is an interval center at 0, i.e., an interval of the form (-R, R) for some  $R \ge 0$  (R is called the **radius of convergence**). The radius of convergence R can be infinite, in which case the interval of convergence is  $(-\infty, \infty)$ .

When a power series converges for every x in an interval (-R, R), it defines a function whose domain is the same interval. The interval of convergence of each of the three series above is  $(-\infty, \infty)$ . If we plug a specific real number, say x = 2 into the first series, then the value of the series  $\sum_{n=0}^{\infty} \frac{2^n}{n!} = 1 + 2 + \frac{4}{2!} + \frac{8}{3!} + \dots$  approaches the value of  $e^2$ 

as more terms are included in the series.

The solutions of many differential equations, especially those with variable coefficients, cannot be expressed explicitly or implicitly in terms of elementary functions. In these situations, we seek series solutions to the equation, that is, we assume a solution in the form of an infinite series and proceed in a way similar to the method of undetermined coefficients to find a pattern of the coefficients  $a_n$  of the series. Before we study this technique, we need to review some basic operations with infinite series.

Exa	m	ple	• <b>1</b> :	Sh	nifti	ng	$\mathbf{the}$	ind	$\mathbf{e}\mathbf{x}$	of s	um	mat	ion																		
													$\infty$																		
The	sy	mb	ol	sign	na i	n th	le po	ower	ser	ies n	otat	ion	$\sum_{i}$	$a_n x$	<sup>n</sup> de	enote	es <i>s</i> a	umn	ratic	on a	nd r	a dei	note	s th	e in	dex	of s	sumr	nati	on,	
whi	ch s	er	ves	as a	a coi	inte	er. Iı	n ma	any :	situa	atior	ns, b	$e^{n=0}$	e we	e car	ı cor	nbir	ne tv	vo oi	: mo	res	umn	nati	$\mathbf{ons}$	as a	sing	gle s	umn	natio	on,	
we i	ieeo	d t	o r		lex nori	or s	hift rm	the	$ind\epsilon$	ex of	sun sor	nma	tion	. In	the $x^n$	folle	owir od (	$\operatorname{ng} \operatorname{ez}$	$\max_{n+2}$	ples.	we $n-2$	will	shi	ft tł	ne ir	ıdex	of	sumi	mati	on	
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1	. 5	$\tilde{\mathbf{x}}$	$x^n$	+2												2.	$\sum_{r=1}^{\infty} r$	n(n -	- 1)	$a_n x^i$	$^{n-2}$ .										
	$\frac{2}{n}$	=0	n	!												2 n	=2			10											

2	solu	ltion																								
	1.	Let $k = n +$	- 2. W	hen n	= 0. k	r = 2	2. Al	so.	n =	k -	2.	So tl	he s	eries	s be	$\cos$	es									
								,																		
											$\infty$	$-x^{\prime}$	c													
											<u> </u>	(k -	2)!													
											c=2															
		Since the in	dex of	i summ	nation	is a	dum	my	para	ame	ter,	we i	rena	me	the	k in	the	abo	ve e	xpre	essic	n to	n a	nd 1	cewr	ite
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											<u> </u>	<i>x</i> <sup>*</sup>	ı													
										1	n=2	(n -	2)!	,												
		which is a	series	whose	o o o o o o o	ric t	erm	inv	olve	$= r^{i}$	n in	stea	d of	$r^n$	+2	On	e ca	n e	asilı	r ch	eck	tha	t th	19 94	ries	is
		equivalent t	o the	origing	al serie	s bv	writ	ing	out	$\frac{3}{a}$	ew t	erm	s an	d oł	oserv	ve th	nat t	hev	are	exa	ctlv	the	san	ne fc	or be	th
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				Ori	oinal s	erie	. <b>V</b>	$\overline{\ } x$	n+2	_ /	$r^2$ +	$\underline{x^3}$	$+ \frac{x}{x}$	.4 +	$\frac{x^5}{2}$	+	(N	ote	that	0! :	= 1)					
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		Note that w	vhen w	ve work	with	infin	ite s	erie	s th	e 11	nnei	· lim	it o	f sur	nme	tior	is r	not a	ffec	ted	bv t	he s	hifti	ing r	proce	ess
		because it a	lwavs	remai	ns infi	nity.		0110	.,	u	ppo.									oou	~ <u>j</u> .				100	
	2.	Let $k = n -$	- 2. W	hen n	= 2, k	c = 0	). Al	so,	n =	k +	-2 a	nd r	ı — ∶	1 =	k +	1. S	o th	e se	ries	bec	ome	s				
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									$\sum$	(k -	+2)	(k +	1)a	k+2	$x^k$ .											
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		$r^n$ instead	$f_{n-1}$	$\frac{n}{2}$ .	get a	serie	:5 WI.	nen	15 6	qui	vale	10 00	) [110		gina	ii se	nes	anu	WII	use	gene	eric	tern	115 11	IVOIN	/es
		a moreau (	JI J	·					$\infty$	,		,			~											
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Example 2: Shifting in	ndices to verify an equality		
$\infty$	<u>∞</u>		
Show that $x^2 \sum_{n \in \mathbb{N}} n(n+1)$	$a_n x^n = \sum_n (n-2)(n-1)a_{n-2}x^n.$		
	n=2		

	Solu	tio	n																										
,	Take	$_{\mathrm{the}}$	$x^2$	insi	de t	he s	umr	nati	on c	n tł	ne le	ft h	and	side	and	l mı	ıltip	ly <i>x</i>	<sup>2</sup> ar	$\operatorname{nd} x$	$^{n},$ v	ve o	btaiı	n					
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										$x^2$	$\sum i$	n(n	+1	$a_n x$	n =	$\sum$	n(n	+1	$)a_n$	$x^{n+2}$	2.								
										1	n=0					n=0	1												
	Let /	c = c	n +	2. V	Whe	n n	= 0	, k =	= 2.	Als	o, n	= k	c – 2	2, n	+1	= k	- 1.	We	e the	en h	ave								
			° V	°,		1)	n	+2	$\nabla^{\infty}$	1		(1	1)		k	$\overline{\nabla}$				1)		n	т	. 1	. 1	1	• 1		
			$\sum_{n=1}^{n}$	$\sum_{i=0}^{n}$	(n +	- 1)a	$n^{n}x$	· =	$= \sum_{k=1}^{n}$	$\frac{\kappa}{2}$	- 2)	(ĸ -	- 1)(	$l_{k-2}$	<i>x</i> =	$\sum_{n=1}^{n}$	$_{2}^{(n)}$	- Z,	(n -	- 1)	$a_{n-2}$	$_2x$	= 1	tign	t na	na s	siae,		
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## Example 3: Adding power series

$\operatorname{Con}$	nbine	e the	ро	wer	serie	es ir	nto a	a sin	gle <sup>.</sup>	pow	er se	eries	wh	ose	gene	ric 1	$\operatorname{term}$	inv	olve	s $x^{\prime}$	ı					
	$\infty$				c	×										$\infty$				$\infty$						
1	$\sum$	$na_r$	$x^{n}$	$^{-1} +$	$-2\sum$	$\sum a$	$x^{n}$	+1							2. 2	$\sum$	$a_n x$	n+1	$+ \sum$	$\sum n$	$b_n x$	n - 1				
	n =	1			n	=0										n=0	)		n	=1						

Solu	tio	n																											
1.	We	e wi	l pe	rfor	m sl	hifti	ng o	of ine	dices	s on	bot	h se	ries	(set	: k =	= n	- 1	for t	he f	irst	and	set	k =	= <i>n</i> -	-1f	or t	he se	$\operatorname{con}$	d)
	an	d co	n bi	ne t	hem	:	0																						
						$\infty$				$\infty$				$\infty$				,	$\infty$		,								
						$\sum_{i=1}^{n}$	$na_n$	$r^{n-1}$	+2	$\sum$	$a_n$	$c^{n+1}$	=	<b>)</b> (	k +	$1)a_k$	$x_{t+1}x$	<sup><i>k</i></sup> +	$\sum$	$2a_{k}$	$-1x^{k}$								
					1	n=1				<i>n=</i> (	)			k=0			$\infty$		k=1			$\infty$							
													=	$1 \cdot a_{1}$	$1 \cdot x'$	• + ]	$\sum ($	k +	$1)a_k$	$x_{+1}x$	$^{k}$ +	$\sum$	$2a_{k}$	$-1x^{k}$	;				
															$\infty$	i	k=1					k=1							
													=	$a_1 + $	$\sum$	(k	+1	$a_{k+}$	$x^k$	+2	$a_{k-1}$	$(x^k)$							
															k=1	-													
													=	$a_1 +$	$\sum$	((k	+1)	$a_{k+}$	1+	$2a_k$ .	_1) a	$c^k$							
															k=1														
													=	$a_1 +$	$\sum_{k=1}^{\infty}$	((n	+1	$a_{n+1}$	1+	$2a_n$	_1)	$x^n$ .							
														1.	$\sum_{n=1}^{n}$			, ,,,	1.	10	1)								
	No	te t	hat	in tl	he fi	rst e	equa	lity,	we	$\operatorname{can}$	not	$\operatorname{com}$	bine	e the	e tw	o sei	ries	beca	use	one	sta	rts a	t $k$	= 0	whe	ereas	s the	oth	er
	on	e sta	arts	at A	k =	1. l	[n tl	ne se	$\operatorname{econ}$	d eo	qual	ity,	we	writ	e th	e fir	st t	$\operatorname{erm}$	of t	he f	irst	seri	es o	utsi	de t	he s	umm	nati	on
	no	tatio	n w	hich	n me	ikes	bot	h se	ries	star	t at	<i>k</i> =	= 1.	Thi	s all	ows	us t	o co	mbi	ne t	hem	i in	the	nex	t ste	p.			
2.	Us	e th	e sa	me s	strat	tegy	as a	abov	ve, w	ve ha	ave																		
						$\infty$				$\infty$			_	$\infty$															
					2	$\Sigma$	$a_n x$	n+1	+	$\sum n$	$b_n x$	n-1	=	$\sum_{i=1}^{2}$	$a_{k-1}$	$x^k$	+ >	(k)	+1)	$b_{k+}$	$x^k$								
						n=0	0		n	=1			k	=1			k=	$\infty$											
						= >	$\sum 2c$	$l_{k-1}$	$x^k +$	$b_1$	$+ \sum$	$\sum (k$	+1	$b_{k+}$	$1x^k$	= b	1+	$\sum$	$(2a_k)$	-1 -	-(k	+1	$b_{k+}$	$_{1}) x$	k				
						k=	=1	$\infty$			k=	=1						k=1											
						$= b_1$	+	$\sum$ (	$2a_{n}$	-1 +	(n	+1)	$b_{n+}$	$(1) x^{3}$	n •														
							n	i=1																					

Example 4: Obtain a recu	ence relation from an identity	
Show that the identity		
	$\sum_{n=1}^{\infty} na x^{n-1} + 2 \sum_{n=1}^{\infty} a x^{n+1} = 0$	
	$\begin{array}{c c} & & & \\ & & & \\ & & & \\ n=1 \end{array} \end{array} \xrightarrow{\begin{subarray}{c} 1 & a \\ n=0 \end{array}} a_n a_n a_n a_n a_n a_n a_n a_n a_n a_n$	
implies that $a_1 = 0$ and $a_{n+1}$	$-\frac{2}{2}a_{n-1}$ for each $n > 1$ .	

#### Solution

In Example 3, we added the two power series on the left hand side of this identity into a single power series. Using this result, the identity is equivalent to

$$a_1 + \sum_{n=0}^{\infty} \left( (n+1)a_{n+1} + 2a_{n-1} \right) x^n = 0.$$

Thus, we have a power series that sums to zero. The only way for this to hold is when each of its coefficients equals zero. It follows that

$$a_1 = 0$$
 and  $(n+1)a_{n+1} + 2a_{n-1} = 0$ , for  $n \ge 1$ .

Getting  $a_{n+1}$  by itself in the second equation gives rise to the **recurrence relation**  $a_{n+1} = -\frac{2}{n+1}a_{n-1}, \text{ for } n \ge 1.$ 

This helps us determine  $a_{n+1}$  in terms of  $a_{n-1}$  for n = 1, 2, 3, ...

#### **Derivatives of Power Series**

If a power series  $\sum_{n=0}^{\infty} a_n x^n$  converges on the interval (-R, R) where R > 0, then it defines a differentiable function f(x) on (-R, R). The power series for the derivatives of f can be found by the process of term-by-term differentiation. More specifically, if  $f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$ , then

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \dots = \sum_{n=1}^{\infty} a_n n x^{n-1} \text{ and}$$
$$f''(x) = 2a_2 + 6a_3x^2 + 12a_4x^2 + \dots = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2}.$$

Note that the index of summation starts at n = 1 for the series that represents f' and at n = 2 for the one that represents f''.

Exa	$\mathbf{mpl}$	e 5	: Ve	erify	/ a s	seri	es s	olut	tion	1														
								8	(_	$1)^{n+1}$	-1													
Veri	y th	at t	he j	powe	er se	ries	y =	$\sum$	<u> </u>	$\frac{1}{n}$	$-x^n$	is a	a so	lutio	on o	f the	e equ	iatio	n					
								<i>n=</i> 0				,		.,		_								
												(x	+1	$)y^{\prime\prime}$	+y'	= 0	).							

Solution	
We have	
$y' = \sum \frac{(-1)^{n+1}}{n} n x^{n+1} = \sum (-1)^{n+1} x^{n-1}$	
$u'' = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n-1)x^{n-2}} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(n-1)x^{n-2}}{n(n-1)x^{n-2}}.$	
Substitute these into the differential equation, we have	
$(x+1)y''+y' = (x+1)\sum_{n=1}^{\infty} (-1)^{n+1}(n-1)x^{n-2} + \sum_{n=1}^{\infty} (-1)^{n+1}x^{n-1}$	
$=\sum_{n=1}^{\infty} (-1)^{n+1} (n-1)x^{n-1} + \sum_{n=1}^{\infty} (-1)^{n+1} (n-1)x^{n-2} + \sum_{n=1}^{\infty} (-1)^{n+1}x^{n-1}$	
$=\sum_{k=1}^{\infty} (-1)^{k+2} k x^{k} + \sum_{k=1}^{\infty} (-1)^{k+3} (k+1) x^{k} + \sum_{k=1}^{\infty} (-1)^{k+2} x^{k}$	
$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$	
$-\sum_{k=1}^{\infty} (-1)^{k} k x^{k} + \sum_{k=1}^{\infty} (-1)^{k} (k+1) x^{k} + \sum_{k=1}^{\infty} (-1)^{k} x^{k}$	
$\begin{array}{c c c c c c c c c c c c c c c c c c c $	
$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	
$= \sum_{k=1}^{n} (-1)^{k} k x^{k} - \sum_{k=0}^{n} (-1)^{k} k x^{k} - \sum_{k=0}^{n} (-1)^{k} x^{k} + \sum_{k=0}^{n} (-1)^{k} x^{k}$	
$= \sum (-1)^k k x^k - 0 - \sum (-1)^k k x^k = 0.$	

### Series Solutions of Differential Equations

To apply the method of power series solution to solve the equation  $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$ , we assume that the equation has a series solution (within the scope of this lecture, we also assume the series is centered at 0) of the form  $y = \sum_{n=0}^{\infty} a_n x^n$ . Substitute this series (and its derivatives) into the given equation, combine series and argue that the coefficients of powers of x must be zero. This gives rise to a recurrence relation among the coefficients  $a_n$  of the series solution.

## Example 6: Power Series Solution

Solve the differential equation y'' + y = 0 assuming a power series solution centered at 0

Sol	utio	n																											
Not	e th	at v	ve ca	an e	easily	y so	lve	$_{\rm this}$	equ	atic	n b	y us	sing	the	cha	$rac^{1}$	terist	ic e	equa	tion	an	d ol	otaiı	n th	e g	ener	al so	oluti	on
y =	$c_1  \mathrm{c}$	$\operatorname{os} x$	$+ c_2$	$\sin \theta$	<i>x</i> . Ľ	et's	see	how	the	e me	tho	d of	seri	es so	oluti	on '	works	s he	ere.										
We	begi	n by	ass	umi	ng a	a po	wer	serie	es so	luti	on c	of th	e fo	$\operatorname{rm} i$	y = 1	$\sum$	$a_n x^n$	. Т	ake	$_{\mathrm{the}}$	deri	vati	$\mathbf{ves}$	of $y$	, we	get			
															7	n=0													
								21		$\sum_{n=1}^{\infty}$		n-1	อท	1 <i>al</i> ''	_ <	$\sum_{n=1}^{\infty}$	(n_	1)	$r^{n}$	n-2									
								9		$\sum_{n=1}^{\prime}$	$u_n$ .		an	чy	$\begin{bmatrix} -2\\ n \end{bmatrix}$	=2	0(10	1)0	$u_n x$	•									
								y'	=	$\sum_{n=1}^{n} p$	ia <sub>n</sub> :	$c^{n-1}$	an	d <i>y''</i>	$=$ $\sum_{n}$	$\sum_{i=2}^{n} r$	n(n-1)	1)	$a_n x'$	<i>i</i> -2.									

Solution		
Substitute $y''$ and $y$ to	the equation, and combine the series, we have	
	$\sum_{n=1}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} a_n x^n = 0$	
	$\sum_{k=1}^{\infty} (k+2)(k+1)a = e^{k} + \sum_{k=1}^{\infty} a = e^{k}$ (Shift the first index)	_
	$\sum_{k=0}^{n} \binom{n+2}{n+1} \binom{n+2}{n+2} + \sum_{n=0}^{n} \binom{n}{n} \binom{n}{n} = 0  (\text{Shift the first fidex})$	_
	$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^{n} + \sum_{n=0}^{\infty} a_n x^n = 0  (\text{Rename the first index})$	
	n=0 $n=0$	
	$\sum \left( (n+2)(n+1)a_{n+2} + a_n \right) x^n = 0  \text{(Combine the two series)}$	
In the last equation,	we have power series that sums to zero. The only way for this to hold is when each of it	.s
coefficients equals zero	It follows that	
	$(n+2)(n+1)a_{n+2} + a_n$ for each $n \ge 0$ .	
<b>TT7</b> (1) 1 (1)		
we then obtain the re	urrence relation	
	$a_{n+2} = -\frac{a_n}{(n+2)(n+1)}, \text{ for } n \ge 0.$	
	(n+2)(n+1)	
Plug in a few values o	the index n, we have	
	$m = 0$ , $a_0 = a_0$	
	$n = 0$ . $a_2 = -\frac{1}{2}$	
	$n = 1$ : $a_3 = -\frac{a_1}{3 \cdot 2} = -\frac{a_1}{3!}$	
	$n - 2$ $a_1 a_2 - a_0 - a_0$	
	$n^{-2} \cdot a_4 - 4 \cdot 3 - 2 \cdot 3 \cdot 4 - 4!$	
	$n = 3$ : $a_5 = -\frac{a_3}{5 \cdot 4} = \frac{a_1}{2 \cdot 3 \cdot 4 \cdot 5} = \frac{a_1}{5!}$	
	$n = 4$ : $a_6 = -\frac{a_4}{a_6} = -\frac{a_0}{a_0} = -\frac{a_0}{a_0}$	
	$6 \cdot 5$ (4!)(5)(6) 6!	
	$n = 5$ : $a_7 = -\frac{a_5}{7.6} = -\frac{a_1}{(5!)(6)(7)} = -\frac{a_1}{7!}$ and so on.	
It follows that the solu	cion to the equation is	
		_
<i>y</i> =	$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^3 + a_6 x^6 + a_7 x^4 + \dots$	
	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	
=	$a_0 + a_1 x - 2! x - 3! x + 4! x + 5! x - 6! x - 7! x + \dots$	
=	$a_0 = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots + a_1 = x - \frac{1}{2!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$	
Thus, the solution $y$ is	a combination $y = a_0y_1 + a_1y_2$ where $y_1$ and $y_2$ are functions with series representation	
	1 - 1 - 1 - 1 - 1 - 1 - 1 - 1 - 1 - 1 -	
9	$-1 - \frac{1}{2!}x + \frac{1}{4!}x - \frac{1}{6!}x + \dots \text{ and } y_2 = x - \frac{1}{3!}x + \frac{1}{5!}x - \frac{1}{7!}x + \dots$	
Observe that these ar	the series representations for the functions $\cos x$ and $\sin x$ , respectively. So, $y_1 = \cos x$ and	d
$y_2 = \sin x$ . And the s	lution to the equation is $y = a_0 \cos x + a_1 \sin x$ which is what we expect. Since there are n	0
initial conditions, $a_0$ a	id $a_1$ play the role of the arbitrary constants.	

# Example 7: Power Series Solution - Airy's equation

Solve the differential equation y'' + xy = 0 assuming a power series solution centered at 0

											$\infty$																
Assume	a po	ower	ser	ies s	olut	ion	of t	he f	$^{ m orm}$	y =	Σ	$a_n$	$c^n$ .	Take	the	de	riva	tives	of	y, w	e ge	t					
	-									Ŭ		) 														_	
									$\infty$						$\infty$												
							y	′ = <sup>`</sup>	$\sum i$	$a_n$	$c^{n-1}$	an	d y'	′ = <b>`</b>	$\sum r$	n(n -	- 1)	$a_n x$	n-2.								
								1	n=1				Ū	r	=2		,										
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			<i></i>		° V	<u>م</u>		1)	n	-2 -	7	×,	- 1														
	0	= y	·· +	xy :	= 2		n -	(1)a	$nx^n$	- +	x 4		$nx^n$														
			$\infty$		n=	=2		0	0		n	,=0															
		= \	$\sum_{n}$	(n -	- 1)	$a_n x^n$	n - 2	$+ \sum$	$\int a_r$	$x^{n+}$	1	(Ta	ke '	the :	c ins	ide	seco	ond	serie	es)							
		n	=2	`	,				=0			Ì								<i>,</i>						_	
		, 7	$\infty$		$\rightarrow$ (1	. 1		k		$\infty$		k	10	1.0	1	. <sub>1</sub> .											
		= _		:+:	$Z)(\kappa$	+ 1	$a_{k}$	$x^{n}$	+ 4		k-1	x	(5	hift	the	indi	ces)										
		ĸ	=0						ĸ	$\infty$																	
		= 5	<b>)</b> (1	i + 1	$\frac{2}{n}$	+ 1	$)a_n$	$+2x^{\prime}$	<sup>1</sup> +	$\sum c$	$l_{n-1}$	$x^n$	(]	Rena	me	the	ind	ices)									
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		= 2	•1•	$a_2$	$\cdot x^{\circ}$	+2	$\sum_{n}^{(n)}$	+ 2	)(n	+1)	$a_{n+}$	$2x^n$	+ 2	$\sum_{i} a_i$	1-13	e''	(P	ull (	out i	nrst	tern	n of	first	se	ries	)	
				$\infty$	)	n=	=1						n	=1													
		= 2	$a_2$ +	- 5	((1	i + i	2)(n	+1	$a_{n}$	$_{-2} +$	$a_{n}$	$_{1}) x$	n														
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me rect	uren	ce n		on					0		·		$a_{n-}$	1		or i		1								_	
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t follov		at tl		of t	ion $\sum_{n=0}^{\infty} a_{n}$	to the transformation $a_1x$	ne e = a + 0	quat $c_0 + \frac{c_0}{2}$	hav n = n = n = n = n = n = $a_1 x$ $a_1 x$ $a_1 x$	e $1$ 2 3 4 5 6 is $+ a_{2}^{3}$ 4 $- a_{2}^{3}$	$\begin{array}{c} \vdots & \epsilon \\ \vdots & \epsilon \\$	$u_3 = u_4 = u_4 = u_5 = u_6 = u_7 = u_8 $	$-\frac{2}{2}$ $-\frac{3}{3}$ $-\frac{4}{5}$ $-\frac{6}{6}$ $-\frac{7}{7}$ +0	$\begin{array}{c} a_0 \\ \hline a_1 \\ \hline a_2 \\ \hline a_2 \\ \hline a_3 \\ \hline a_4 \\ \hline a_5 \\ \hline a_4 \\ \hline a_5 \\ \hline$	$= 0$ $= \frac{1}{2}$ $= 0$ $x^{4} + \frac{1}{3}$ $= 0$	$\begin{array}{c} a_{0} \\ \cdot 3 \\ a_{1} \\ \cdot 4 \\ \cdot a_{1} \\ \cdot 4 \\ \cdot a_{2} \\ \cdot a_{5} \\ \cdot a_{$	$5 \cdot 6$ $6 \cdot 7$ $5 \cdot 6$ 7 $5 \cdot 6$	$a_{6}a$	$a_1$	$a_7 x$ $6 \cdot 7$	7 + 0 x <sup>7</sup> + 0	$a_8x^8$	· · · · ·				
		at tl		of the second s	ion $a = 0$ a = 0 a + 0	to t $a_1x$	ne e = a + 0	quation $x^3 \rightarrow x^3$	hav n = n = n = n = n = $a_1 x$ $a_1 x$	e $1$ 2 3 4 5 6 is $+ a_{2}^{3}$ - 1 1 2 2	$\begin{array}{c} \vdots & \epsilon \\ \hline \vdots & \epsilon \\ \hline \vdots & \epsilon \\ \hline \\ 2x^2 \\ \hline \\ \frac{a_1}{3 \cdot \epsilon} \\ \hline \\ \hline \end{array}$	$u_3 = u_4 = u_5 = u_5 = u_6 = u_7 = u_8 = u_7 = u_8 $	$-\frac{2}{2}$ $-\frac{3}{4}$ $-\frac{4}{5}$ $-\frac{6}{7}$ +0	$\begin{array}{c} a_0 \\ \hline \cdot 3 \\ a_1 \\ \hline \cdot 4 \\ a_2 \\ \hline \cdot 5 \\ \hline \cdot 6 \\ a_4 \\ \hline \cdot 7 \\ a_5 \\ \hline \cdot 8 \\ \hline \cdot 8 \\ \hline + a_4 \\ + \\ - \\ 2 \\ \hline + \\ 2 \\ \hline + \\ + \\ 2 \\ \hline + \\ + \\ - \\ 2 \\ \hline + \\ + \\ - \\ 2 \\ \hline + \\ + \\ - \\ 2 \\ \hline + \\ + \\ - \\ 2 \\ \hline + \\ + \\ - \\ 2 \\ \hline + \\ + \\ - \\ - \\ - \\ + \\ - \\ - \\ - \\ - \\$	$= 0$ $= \frac{1}{2}$ $= 0$ $x^{4} + \frac{1}{a_{0}}$ $= 3 \cdot \frac{1}{a_{1}}$	$a_{0}$ $a_{1}$ $a_{1}$ $a_{2}$ $a_{3}$ $a_{4}$ $a_{5}$ $a_{5}$ $a_{5}$ $a_{6}$ $a_{7}$ $a_{7$	$5 \cdot 6$ $6 \cdot 7$ $5 \circ 6$ $r^5 + r^6 \cdot 1$	$a_{62}$			7 + 0 x7 +	$a_8x^8$	· · · ·				
ft follov	zs th	at tl		of the second s	ion a = 0 a = 0 a = 0 a = 0 a = 0	to the transformation $a_1x$	$ne e = a + 0$ $1$ $2 \cdot 3$	quation $\frac{1}{2}$	hav n = n = n = n = n = 1	e $1$ $2$ $3$ $4$ $5$ $6$ $1$ $1$ $3 - 1$ $3 \cdot 5$ $1$ $1$ $3$ $1$ $1$ $3$ $1$ $1$ $3$ $1$ $1$ $3$ $1$ $1$ $1$ $3$ $1$ $1$ $1$ $1$ $1$ $1$ $1$ $1$ $1$ $1$	$\begin{array}{c} \vdots & \epsilon \\ \vdots & \epsilon \\$	$u_3 = u_4 = u_5 = u_5 = u_6 = u_7 = u_8 = u_8 = u_7 = u_8 $	$-\frac{2}{2}$ $-\frac{3}{3}$ $-\frac{4}{5}$ $-\frac{6}{7}$ +0	$\begin{array}{c} a_0 \\ \cdot 3 \\ a_1 \\ \cdot 4 \\ a_2 \\ \cdot 5 \\ \cdot 6 \\ a_4 \\ \cdot 7 \\ a_5 \\ \cdot 8 \\$	$= 0$ $= \frac{1}{2}$ $= 0$ $= 0$ $x^{4} + \frac{a_{0}}{3}$ $= a_{1}$	$a_{0}$ $a_{1}$ $a_{1}$ $a_{1}$ $a_{2}$ $a_{3}$ $a_{4}$ $a_{5}$ $a_{5}$ $a_{5}$	$5 \cdot 6$ $6 \cdot 7$ so c $x^5 + x^6 - 1$ 3	$a_{62}$		$a_7x$ $6\cdot 7$ $3\cdot 4$	7 + 6	$a_8x^8$					
		at tl		of t	ion $\sum_{n=0}^{\infty} a^{n}$	to t] $a_1 x$	$ne e = a + 0$ $\frac{1}{2 \cdot 3}$	quation $x^3 + \frac{1}{2}$	hav n = n = n = n = n = n = 1 = n = 1 = $2 \cdot$ y =	e $1$ 2 3 4 5 6 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1	$\begin{array}{c} \vdots & \epsilon \\ \vdots & \epsilon \\$	$u_3 = u_4 = u_5 = u_5 = u_6 = u_7 = u_8 $	$-\frac{2}{2}$ $-\frac{3}{3}$ $-\frac{4}{5}$ $-\frac{6}{5}$ $-\frac{7}{7}$ $+ 0$	$\begin{array}{c} a_0 \\ \cdot 3 \\ a_1 \\ \cdot 4 \\ a_2 \\ \cdot 5 \\ \cdot 6 \\ a_3 \\ \cdot 6 \\ a_4 \\ \cdot 7 \\ \cdot 8 \\$	$= 0$ $= \frac{1}{2}$ $= \frac{1}{3}$ $= 0$ $x^{4} + \frac{1}{a_{0}}$ $a_{1} - \frac{1}{a_{1}}$	$a_{0}$ $a_{1}$ $a_{1}$ $a_{1}$ $a_{1}$ $a_{2}$ $a_{3}$ $a_{5}$ $a_{5}$ $a_{5}$ $a_{5}$ $a_{5}$ $a_{5}$	$5 \cdot 6$ $5 \cdot 6$ $6 \cdot 7$ $50 \cdot 6$ $x^{5} + $ $x^{6} - $ $3 \cdot $	$\frac{1}{4}x^{2}$	$a^{6} + a_{1}$	$a_7x$ $6\cdot 7$ $3\cdot 4$ $y_2$	7 + 6	$a_8x^8$	· · · · ·				
t follov		at tl		of t	ion $a = 0$	to the transformation $a_1 x$	$ne e = a + 0$ $1$ $2 \cdot 3$ $a co$	quation $\frac{1}{2}$ and $\frac{1}{2$	hav n = n = n = n = n = n = n = 1 = n = $2 \cdot$ y = y	e 1 2 3 4 5 6 is is 3 - is 3 - is	$\begin{array}{c} \vdots & \epsilon \\ \vdots & \epsilon \\$	$u_3 = u_4 = u_5 = u_5 = u_6 = u_7 = u_8 = u_7 = u_8 $	$-\frac{2}{2}$ $-\frac{3}{3}$ $-\frac{4}{5}$ $-\frac{6}{6}$ $-\frac{7}{7}$ +0 + c	$\begin{array}{c} a_0 \\ \hline a_1 \\ \hline a_2 \\ \hline a_2 \\ \hline a_3 \\ \hline a_4 \\ \hline a_5 \\ \hline a_4 \\ \hline a_5 \\ \hline$	$= 0$ $= \frac{-2}{3}$ $= 0$ $x^{4} + \frac{a_{0}}{3 \cdot 3 \cdot 3}$ $a_{1}$ whe	$a_{0}$ $a_{1}$ $a_{1}$ $a_{1}$ $a_{1}$ $a_{1}$ $a_{1}$ $a_{1}$ $a_{1}$ $a_{2}$ $a_{3}$ $a_{5}$ $a_{5$	$5 \cdot 6$ $5 \cdot 6$ 5	$\frac{1}{4}$	$a_1$	$a_7x$ $6\cdot 7$ $3\cdot 4$ $y_2$ first	$7^7 + 6$ $x^7 + 1$ $1 \cdot 6 \cdot$ 5 few	$a_8x^8$					