Power Series Solutions of Linear Differential Equations

Review of Power Series

A **power series** centered at the point x_0 is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \dots,$$

where x is a variable and the a_n are real numbers called the coefficients of the series. Throughout this lecture, we assume the center is 0, i.e., $x_0 = 0$. A power series centered at 0 takes the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots.$$

The power series representation for some familiar functions are:

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \\ \sin(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \\ \cos(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \end{aligned}$$

We say that a power series $\sum_{n=0}^{\infty} a_n x^n$ **converges** at a real number, say $x = 2$ if when we replace x by 2, the infinite series of real number $\sum_{n=0}^{\infty} a_n 2^n$ converges. Every power series has an **interval of convergence** which is the set of all real numbers x for which the series converges. The interval of convergence of a power series $\sum_{n=0}^{\infty} a_n x^n$ is an interval center at 0, i.e., an interval of the form $(-R, R)$ for some $R \geq 0$ (R is called the **radius of convergence**). The radius of convergence R can be infinite, in which case the interval of convergence is $(-\infty, \infty)$.
When a power series converges for every x in an interval $(-R, R)$, it defines a function whose domain is the same interval. The interval of convergence of each of the three series above is $(-\infty, \infty)$. If we plug a specific real number, say $x = 2$ into the first series, then the value of the series $\sum_{n=0}^{\infty} \frac{2^n}{n!} = 1 + 2 + \frac{4}{2!} + \frac{8}{3!} + \dots$ approaches the value of e^2 as more terms are included in the series.
The solutions of many differential equations, especially those with variable coefficients, cannot be expressed explicitly or implicitly in terms of elementary functions. In these situations, we seek series solutions to the equation, that is, we assume a solution in the form of an infinite series and proceed in a way similar to the method of undetermined coefficients to find a pattern of the coefficients a_n of the series. Before we study this technique, we need to review some basic operations with infinite series.

Example 1: Shifting the index of summation

The symbol *sigma* in the power series notation $\sum_{n=0}^{\infty} a_n x^n$ denotes *summation* and n denotes the *index of summation*, which serves as a counter. In many situations, before we can combine two or more summations as a single summation, we need to *reindex* or *shift the index of summation*. In the following examples, we will shift the index of summation such that the generic term of the power series involves x^n instead of x^{n+2} or x^{n-2}.

1. $\sum_{n=0}^{\infty} \frac{x^{n+2}}{n!}$
2. $\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$.

Solution

1. Let $k = n+2$. When $n = 0$, $k = 2$. Also, $n = k-2$. So the series becomes

$$\sum_{k=2}^{\infty} \frac{x^k}{(k-2)!}.$$

Since the index of summation is a dummy parameter, we rename the k in the above expression to n and rewrite it as

$$\sum_{n=2}^{\infty} \frac{x^n}{(n-2)!},$$

which is a series whose generic term involves x^n instead of x^{n+2}. One can easily check that this series is equivalent to the original series by writing out a few terms and observe that they are exactly the same for both series:

$$\text{Original series: } \sum_{n=0}^{\infty} \frac{x^{n+2}}{n!} = x^2 + \frac{x^3}{1!} + \frac{x^4}{2!} + \frac{x^5}{3!} + \ldots \text{ (Note that } 0! = 1)$$

$$\text{Reindexed series: } \sum_{n=2}^{\infty} \frac{x^n}{(n-2)!} = x^2 + \frac{x^3}{1!} + \frac{x^4}{2!} + \frac{x^5}{3!} + \ldots$$

Note that when we work with infinite series, the upper limit of summation is not affected by the shifting process because it always remains infinity.

2. Let $k = n-2$. When $n = 2$, $k = 0$. Also, $n = k+2$ and $n-1 = k+1$. So the series becomes

$$\sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2} x^k.$$

Rename the k to n, we get a series which is equivalent to the original series and whose generic terms involves x^n instead of x^{n-2}:

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n.$$

Example 2: Shifting indices to verify an equality

Show that $x^2 \sum_{n=0}^{\infty} n(n+1)a_n x^n = \sum_{n=2}^{\infty} (n-2)(n-1)a_{n-2} x^n$.

Solution

Take the x^2 inside the summation on the left hand side and multiply x^2 and x^n, we obtain

$$x^2 \sum_{n=0}^{\infty} n(n+1)a_n x^n = \sum_{n=0}^{\infty} n(n+1)a_n x^{n+2}.$$

Let $k = n+2$. When $n = 0$, $k = 2$. Also, $n = k-2$, $n+1 = k-1$. We then have

$$\sum_{n=0}^{\infty} n(n+1)a_n x^{n+2} = \sum_{k=2}^{\infty}(k-2)(k-1)a_{k-2}x^k = \sum_{n=2}^{\infty}(n-2)(n-1)a_{n-2}x^n = \text{ Right hand side,}$$

where in the last step we rename the index of summation from k to n

Example 3: Adding power series

Combine the power series into a single power series whose generic term involves x^n

1. $\displaystyle\sum_{n=1}^{\infty} na_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^{n+1}$
2. $\displaystyle 2\sum_{n=0}^{\infty} a_n x^{n+1} + \sum_{n=1}^{\infty} nb_n x^{n-1}$

Solution

1. We will perform shifting of indices on both series (set $k = n-1$ for the first and set $k = n+1$ for the second) and combine them:

$$\begin{aligned}
\sum_{n=1}^{\infty} na_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^{n+1} &= \sum_{k=0}^{\infty}(k+1)a_{k+1}x^k + \sum_{k=1}^{\infty} 2a_{k-1}x^k \\
&= 1\cdot a_1 \cdot x^0 + \sum_{k=1}^{\infty}(k+1)a_{k+1}x^k + \sum_{k=1}^{\infty} 2a_{k-1}x^k \\
&= a_1 + \sum_{k=1}^{\infty}\left((k+1)a_{k+1}x^k + 2a_{k-1}x^k\right) \\
&= a_1 + \sum_{k=1}^{\infty}\left((k+1)a_{k+1} + 2a_{k-1}\right)x^k \\
&= a_1 + \sum_{n=1}^{\infty}\left((n+1)a_{n+1} + 2a_{n-1}\right)x^n.
\end{aligned}$$

   Note that in the first equality, we cannot combine the two series because one starts at $k = 0$ whereas the other one starts at $k = 1$. In the second equality, we write the first term of the first series outside the summation notation which makes both series start at $k = 1$. This allows us to combine them in the next step.

2. Use the same strategy as above, we have

$$\begin{aligned}
&2\sum_{n=0}^{\infty} a_n x^{n+1} + \sum_{n=1}^{\infty} nb_n x^{n-1} = \sum_{k=1}^{\infty} 2a_{k-1}x^k + \sum_{k=0}^{\infty}(k+1)b_{k+1}x^k \\
&= \sum_{k=1}^{\infty} 2a_{k-1}x^k + b_1 + \sum_{k=1}^{\infty}(k+1)b_{k+1}x^k = b_1 + \sum_{k=1}^{\infty}\left(2a_{k-1} + (k+1)b_{k+1}\right)x^k \\
&= b_1 + \sum_{n=1}^{\infty}\left(2a_{n-1} + (n+1)b_{n+1}\right)x^n.
\end{aligned}$$

Example 4: Obtain a recurrence relation from an identity

Show that the identity

$$\sum_{n=1}^{\infty} na_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

implies that $a_1 = 0$ and $a_{n+1} = -\dfrac{2}{n+1} a_{n-1}$ for each $n \geq 1$.

Solution

In Example 3, we added the two power series on the left hand side of this identity into a single power series. Using this result, the identity is equivalent to

$$a_1 + \sum_{n=1}^{\infty} \left((n+1)a_{n+1} + 2a_{n-1}\right) x^n = 0.$$

Thus, we have a power series that sums to zero. The only way for this to hold is when each of its coefficients equals zero. It follows that

$$a_1 = 0 \text{ and } (n+1)a_{n+1} + 2a_{n-1} = 0, \text{ for } n \geq 1.$$

Getting a_{n+1} by itself in the second equation gives rise to the **recurrence relation**

$$a_{n+1} = -\frac{2}{n+1} a_{n-1}, \text{ for } n \geq 1.$$

This helps us determine a_{n+1} in terms of a_{n-1} for $n = 1, 2, 3, \ldots$.

Derivatives of Power Series

If a power series $\sum_{n=0}^{\infty} a_n x^n$ converges on the interval $(-R, R)$ where $R > 0$, then it defines a differentiable function $f(x)$ on $(-R, R)$. The power series for the derivatives of f can be found by the process of term-by-term differentiation. More specifically, if $f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots$, then

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots = \sum_{n=1}^{\infty} a_n n x^{n-1} \text{ and}$$

$$f''(x) = 2a_2 + 6a_3 x^2 + 12a_4 x^2 + \cdots = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}.$$

Note that the index of summation starts at $n = 1$ for the series that represents f' and at $n = 2$ for the one that represents f''.

Example 5: Verify a series solution

Verify that the power series $y = \sum_{n=0}^{\infty} \dfrac{(-1)^{n+1}}{n} x^n$ is a solution of the equation

$$(x+1)y'' + y' = 0.$$

Solution

We have

$$y' = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} n x^{n-1} = \sum_{n=1}^{\infty} (-1)^{n+1} x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n} n(n-1) x^{n-2} = \sum_{n=2}^{\infty} (-1)^{n+1} (n-1) x^{n-2}.$$

Substitute these into the differential equation, we have

$$\begin{aligned}
(x+1)y'' + y' &= (x+1)\sum_{n=2}^{\infty}(-1)^{n+1}(n-1)x^{n-2} + \sum_{n=1}^{\infty}(-1)^{n+1}x^{n-1} \\
&= \sum_{n=2}^{\infty}(-1)^{n+1}(n-1)x^{n-1} + \sum_{n=2}^{\infty}(-1)^{n+1}(n-1)x^{n-2} + \sum_{n=1}^{\infty}(-1)^{n+1}x^{n-1} \\
&= \sum_{k=1}^{\infty}(-1)^{k+2}kx^{k} + \sum_{k=0}^{\infty}(-1)^{k+3}(k+1)x^{k} + \sum_{k=0}^{\infty}(-1)^{k+2}x^{k} \\
&= \sum_{k=1}^{\infty}(-1)^{k}kx^{k} + \sum_{k=0}^{\infty}(-1)\left((-1)^{k}(k+1)x^{k}\right) + \sum_{k=0}^{\infty}(-1)^{k}x^{k} \\
&= \sum_{k=1}^{\infty}(-1)^{k}kx^{k} - \sum_{k=0}^{\infty}(-1)^{k}kx^{k} - \sum_{k=0}^{\infty}(-1)^{k}x^{k} + \sum_{k=0}^{\infty}(-1)^{k}x^{k} \\
&= \sum_{k=1}^{\infty}(-1)^{k}kx^{k} - 0 - \sum_{k=1}^{\infty}(-1)^{k}kx^{k} = 0.
\end{aligned}$$

Series Solutions of Differential Equations

To apply the method of power series solution to solve the equation $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$, we assume that the equation has a series solution (within the scope of this lecture, we also assume the series is centered at 0) of the form $y = \sum_{n=0}^{\infty} a_n x^n$. Substitute this series (and its derivatives) into the given equation, combine series and argue that the coefficients of powers of x must be zero. This gives rise to a recurrence relation among the coefficients a_n of the series solution.

Example 6: Power Series Solution

Solve the differential equation $y'' + y = 0$ assuming a power series solution centered at 0

Solution

Note that we can easily solve this equation by using the characteristic equation and obtain the general solution $y = c_1 \cos x + c_2 \sin x$. Let's see how the method of series solution works here.

We begin by assuming a power series solution of the form $y = \sum_{n=0}^{\infty} a_n x^n$. Take the derivatives of y, we get

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \text{ and } y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Solution

Substitute y'' and y to the equation, and combine the series, we have

$$\begin{aligned}
\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n &= 0 \\
\sum_{k=0}^{\infty} (k+2)(k+1)a_{n+2} x^{k} + \sum_{n=0}^{\infty} a_n x^n &= 0 \quad \text{(Shift the first index)} \\
\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^{n} + \sum_{n=0}^{\infty} a_n x^n &= 0 \quad \text{(Rename the first index)} \\
\sum_{n=0}^{\infty} \left((n+2)(n+1)a_{n+2} + a_n\right) x^n &= 0 \quad \text{(Combine the two series)}
\end{aligned}$$

In the last equation, we have power series that sums to zero. The only way for this to hold is when each of its coefficients equals zero. It follows that

$$(n+2)(n+1)a_{n+2} + a_n \text{ for each } n \geq 0.$$

We then obtain the recurrence relation

$$a_{n+2} = -\frac{a_n}{(n+2)(n+1)}, \text{ for } n \geq 0.$$

Plug in a few values of the index n, we have

$$\begin{aligned}
n=0 &: \quad a_2 = -\frac{a_0}{2} \\
n=1 &: \quad a_3 = -\frac{a_1}{3\cdot 2} = -\frac{a_1}{3!} \\
n=2 &: \quad a_4 = -\frac{a_2}{4\cdot 3} = \frac{a_0}{2\cdot 3\cdot 4} = \frac{a_0}{4!} \\
n=3 &: \quad a_5 = -\frac{a_3}{5\cdot 4} = \frac{a_1}{2\cdot 3\cdot 4\cdot 5} = \frac{a_1}{5!} \\
n=4 &: \quad a_6 = -\frac{a_4}{6\cdot 5} = -\frac{a_0}{(4!)(5)(6)} = -\frac{a_0}{6!} \\
n=5 &: \quad a_7 = -\frac{a_5}{7\cdot 6} = -\frac{a_1}{(5!)(6)(7)} = -\frac{a_1}{7!} \text{ and so on.}
\end{aligned}$$

It follows that the solution to the equation is

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + \dots \\
&= a_0 + a_1 x - \frac{a_0}{2!}x^2 - \frac{a_1}{3!}x^3 + \frac{a_0}{4!}x^4 + \frac{a_1}{5!}x^5 - \frac{a_0}{6!}x^6 - \frac{a_1}{7!}x^7 + \dots \\
&= a_0 \left(\underbrace{1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots}_{y_1} \right) + a_1 \left(\underbrace{x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots}_{y_2} \right)
\end{aligned}$$

Thus, the solution y is a combination $y = a_0 y_1 + a_1 y_2$ where y_1 and y_2 are functions with series representation

$$y_1 = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots \text{ and } y_2 = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$$

Observe that these are the series representations for the functions $\cos x$ and $\sin x$, respectively. So, $y_1 = \cos x$ and $y_2 = \sin x$. And the solution to the equation is $y = a_0 \cos x + a_1 \sin x$ which is what we expect. Since there are no initial conditions, a_0 and a_1 play the role of the arbitrary constants.

Example 7: Power Series Solution - Airy's equation

Solve the differential equation $y'' + xy = 0$ assuming a power series solution centered at 0

Solution

Assume a power series solution of the form $y = \sum_{n=0}^{\infty} a_n x^n$. Take the derivatives of y, we get

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \text{ and } y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Substitute y'' and y to the equation, and combine the series, we have

$$\begin{aligned}
0 = y'' + xy &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+1} \quad \text{(Take the } x \text{ inside second series)} \\
&= \sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2} x^k + \sum_{k=1}^{\infty} a_{k-1} x^k \quad \text{(Shift the indices)} \\
&= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=1}^{\infty} a_{n-1} x^n \quad \text{(Rename the indices)} \\
&= 2 \cdot 1 \cdot a_2 \cdot x^0 + \sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=1}^{\infty} a_{n-1} x^n \quad \text{(Pull out first term of first series)} \\
&= 2a_2 + \sum_{n=1}^{\infty} \left((n+2)(n+1)a_{n+2} + a_{n-1}\right) x^n
\end{aligned}$$

Since the last series sums to 0, we must have $a_2 = 0$ and $(n+2)(n+1)a_{n+2} + a_{n-1} = 0$ for $n \geq 1$. This gives rise to the recurrence relation

$$a_{n+2} = -\frac{a_{n-1}}{(n+1)(n+2)}, \text{ for } n \geq 1.$$

Plug in a few values of the index n, we have

$$\begin{aligned}
n = 1 \quad &: \quad a_3 = -\frac{a_0}{2 \cdot 3} \\
n = 2 \quad &: \quad a_4 = -\frac{a_1}{3 \cdot 4} \\
n = 3 \quad &: \quad a_5 = -\frac{a_2}{4 \cdot 5} = 0 \\
n = 4 \quad &: \quad a_6 = -\frac{a_3}{5 \cdot 6} = \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6} \\
n = 5 \quad &: \quad a_7 = -\frac{a_4}{6 \cdot 7} = \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7} \\
n = 6 \quad &: \quad a_8 = -\frac{a_5}{7 \cdot 8} = 0 \text{ and so on.}
\end{aligned}$$

It follows that the solution to the equation is

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 \ldots \\
&= a_0 + a_1 x + 0 - \frac{a_0}{2 \cdot 3} x^3 - \frac{a_1}{3 \cdot 4} x^4 + 0 + \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6} x^6 + \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7} x^7 + 0 + \ldots \\
&= a_0 \left(\underbrace{1 - \frac{1}{2 \cdot 3} x^3 + \frac{1}{2 \cdot 3 \cdot 5 \cdot 6} x^6 - \ldots}_{y_1} \right) + a_1 \left(\underbrace{x - \frac{1}{3 \cdot 4} x^4 + \frac{1}{3 \cdot 4 \cdot 6 \cdot 7} x^7 - \ldots}_{y_2} \right)
\end{aligned}$$

Thus, the general solution y is a combination $y = a_0 y_1 + a_1 y_2$ where we have the first few terms of the power series representation of y_1 and y_2 above. The functions y_1 and y_2 are not elementary functions but we can use the process to generate as many terms of the series representations of them as we'd like.