Definition of the Laplace Transform

Review of Improper Integrals

Let $f(x)$ be a continuous function on the interval $[0, \infty)$, the improper integral $\int_0^{\infty} f(x)dx$ is defined as

$$\int_0^{\infty} f(x)dx = \lim_{b\to\infty} \int_0^{b} f(x)dx.$$

If the limit exists, we say that the integral **converges**; otherwise, we say that the integral **diverges**.

Example 1: Evaluate Improper Integrals

Determine whether the integral converges or diverges. Evaluate the integral if it converges.

1. $\displaystyle\int_0^{\infty} \frac{1}{1+x}dx$
2. $\displaystyle\int_0^{\infty} \frac{1}{1+x^2}dx$

Solution

1. $\displaystyle\int_0^{\infty} \frac{1}{1+x}dx = \lim_{b\to\infty} \int_0^{b} \frac{1}{1+x}dx = \lim_{b\to\infty} (\ln(b+1) - \ln(1)).$

   Since $\ln(b+1) \longrightarrow \infty$ as $b \longrightarrow \infty$, the integral diverges.

2. $\displaystyle\int_0^{\infty} \frac{1}{1+x^2} = \lim_{b\to\infty} \int_0^{b} \frac{1}{1+x^2} = \lim_{b\to\infty} (\arctan(b) - \arctan(0)) = \frac{\pi}{2}.$

   Hence, the integral converges to $\pi/2$.

Laplace Transform

Let $f(x)$ be a function defined on the interval $[0, \infty)$. The **Laplace transform** of f is a function F given by the formula

$$F(s) = \mathscr{L}\{f(x)\} = \int_0^{\infty} e^{-sx} f(x)dx.$$

The domain of the function $F(s)$ is all the values of s for which the integral converges. The Laplace transform of f is denoted by both F and $\mathscr{L}\{f\}$. The Laplace transform is an operator which takes a function f (in the variable x) and transform into another function F (in the variable s). This is an example of an integral operator.

Example 2: Find the Laplace transform of a function by definition

Find the Laplace transform of the constant function $f(x) = 1$, $x \geq 0$.

Solution

By definition of the Laplace transform, we have

$$\begin{aligned} F(s) = \mathscr{L}\{1\} &= \int_0^\infty e^{-sx} \cdot 1dx = \lim_{b\to\infty} \int_0^b e^{-sx} dx \\ &= \lim_{b\to\infty} \frac{-e^{-sx}}{s}\Big|_0^b = \lim_{b\to\infty} \left[-\frac{e^{-sb}}{s} + \frac{1}{s}\right]. \end{aligned}$$

For $s > 0$ and fixed, the exponent $-sb$ is negative so $e^{-sb} \longrightarrow 0$ as $b \longrightarrow \infty$. Hence, the last limit equals $\frac{1}{s}$. Thus,

$$F(s) = \frac{1}{s} \text{ when } s > 0.$$

When $s < 0$, we have $e^{-sb} \longrightarrow \infty$ as $b \longrightarrow \infty$, so the integral diverges. When $s = 0$, the integral $\int_0^\infty e^{-sx} dx = \int_0^\infty 1dx$ diverges as well.

Thus, the Laplace transform of $f(x) = 1$, $x \geq 0$ is the function $F(s) = \frac{1}{s}$ with the domain $s > 0$.

Example 3: Find the Laplace transform of a function by definition

1. Let a be a constant. Find the Laplace transform of the function $f(x) = e^{ax}$, $x \geq 0$.
2. Use your result from part 1 to obtain $\mathscr{L}\{e^{5x}\}$ and $\mathscr{L}\{e^{-8x}\}$.

Solution

1. By definition of the Laplace transform, we have

$$\begin{aligned} F(s) = \mathscr{L}\{e^{ax}\} &= \int_0^\infty e^{-sx} e^{ax} dx = \int_0^\infty e^{-(s-a)x} dx \\ &= \lim_{b\to\infty} \int_0^b e^{-(s-a)x} dx = \lim_{b\to\infty} \frac{-e^{-(s-a)x}}{s-a}\Big|_0^b \\ &= \lim_{b\to\infty} \left[-\frac{e^{-(s-a)x}}{s-a} + \frac{1}{s-a}\right]. \end{aligned}$$

   By the same reasoning as the previous example, the integral converges when $s - a > 0$ and it converges to $\frac{1}{s-a}$. The integral diverges when $s - a \leq 0$. Thus, the Laplace transform of $f(x) = e^{ax}$, $x \geq 0$ is the function $F(s) = \frac{1}{s-a}$ with the domain $s > a$.

2. By part 1, we have $\mathscr{L}\{e^{5x}\} = \frac{1}{s-5}$ with domain $s > 5$. And $\mathscr{L}\{e^{-8x}\} = \frac{1}{s+8}$ with domain $s > -8$.

Example 4: Laplace transform of a piecewise function by definition

Find the Laplace transform of the function

$$f(x) = \begin{cases} 1 & 0 \leq x < 2 \\ x & x \geq 2 \end{cases}$$

Solution

Since $f(x)$ is defined by different formulas on different intervals, we need to break up the integral in the definition of the Laplace transform of f into different parts, we have

$$\begin{aligned}
F(s) &= \int_0^\infty e^{-sx} f(x)dx \\
&= \int_0^2 e^{-sx}dx + \int_2^\infty xe^{-sx}dx \\
&= \frac{-e^{-sx}}{s}\Big|_0^2 - \lim_{b\to\infty}\left(\frac{e^{-sx}}{s^2}+\frac{xe^{-sx}}{s}\right)\Big|_2^b \quad \text{(Integration by parts for the second integral)} \\
&= -\frac{e^{-2s}}{s} + \frac{1}{s} + \frac{e^{-2s}}{s^2} + \frac{2e^{-2s}}{s} \quad \text{when } s>0 \\
&= \frac{1}{s} + \frac{e^{-2s}}{s^2} + \frac{e^{-2s}}{s} \text{when } s>0.
\end{aligned}$$

Note that in from Step 3 to Step 4, we have used the fact that $\lim\limits_{b\to\infty} be^{-sb} = 0$ when $s > 0$ (L'Hopital Rule).

Laplace Transforms of Basic Functions

Using the definition of the Laplace transform, we can obtain the Laplace transforms of many basic functions.

Function $f(x)$	Laplace Transform $F(s) = \mathscr{L}\{f(x)\}$
1	$\frac{1}{s}$, $s>0$
e^{ax}	$\frac{1}{s-a}$, $s>a$
x^n, $n=1,2,\dots$	$\frac{n!}{s^{n+1}}$, $s>0$
$\sin(bx)$	$\frac{b}{s^2+b^2}$, $s>0$
$\cos(bx)$	$\frac{s}{s^2+b^2}$, $s>0$
$\sinh(bx)$	$\frac{b}{s^2-b^2}$, $s>b$
$\cosh(bx)$	$\frac{s}{s^2-b^2}$, $s>b$

Linearity of the Laplace Transform

If we know $\mathscr{L}\{f(x)\} = F(s)$ and $\mathscr{L}\{g(x)\} = G(s)$, then the Laplace transform of the function $h(x) = 2f(x) + 3g(x)$ will be $\mathscr{L}\{h(x)\} = 2F(s) + 3G(s)$. More generally, if f and g are functions whose Laplace transforms exist for $s > a$ and c_1, c_2 are constants, then for $s > a$, we have

$$\mathscr{L}\{c_1 f(x) + c_2 g(x)\} = c_1\mathscr{L}\{f(x)\} + c_2\mathscr{L}\{g(x)\}.$$

This is called the linearity property of the Laplace transform. In other words, the Laplace transform is a **linear operator**.

Example 5: Find Laplace Transforms of functions using Table and Linearity

Use the table of basic Laplace transforms and linearity property to find the Laplace transform of the function:

1. $\mathscr{L}\{5 - e^{2x} + 6x^2\}$
2. $\mathscr{L}\{\cos(5x) + \sin(2x)\}$

Solution

1. For $s > 2$, we have

$$\begin{aligned}\mathscr{L}\{5 - e^{2x} + 6x^2\} &= 5\mathscr{L}\{1\} - \mathscr{L}\{e^{2x}\} + 6\mathscr{L}\{x^2\} \\ &= 5 \cdot \frac{1}{s} - \frac{1}{s-2} + 6 \cdot \frac{2!}{s^3} = \frac{5}{s} - \frac{1}{s-2} + \frac{12}{s^3}.\end{aligned}$$

2. For $s > 0$, we have

$$\mathscr{L}\{\cos(5x) + \sin(2x)\} = \mathscr{L}\{\cos(5x)\} + \mathscr{L}\{\sin(2x)\} = \frac{s}{s^2+25} + \frac{2}{s^2+4}.$$

Example 6: Find Laplace Transforms of functions using Table and Linearity

Use the table of basic Laplace transforms and linearity property to find the Laplace transform of the function:

1. $\mathscr{L}\{e^{-x}\cosh(x)\}$
2. $\mathscr{L}\{\cos^2 x\}$

Solution

Before we find the Laplace transform, we need to rewrite these functions

1. We have:

$$\begin{aligned}\mathscr{L}\{e^{-x}\cosh(x)\} &= \mathscr{L}\{e^{-x} \cdot \frac{e^x + e^{-x}}{2}\} \\ &= \mathscr{L}\left\{\frac{1 + e^{-2x}}{2}\right\} = \frac{1}{2}\left(\mathscr{L}\{1\} + \mathscr{L}\{e^{-2x}\}\right) \\ &= \frac{1}{2}\left(\frac{1}{s} + \frac{1}{s+2}\right), \text{ for } s > 0.\end{aligned}$$

2. Using the trig identity $\cos^2 x = \dfrac{1 + \cos(2x)}{2}$

$$\begin{aligned}\mathscr{L}\{\cos^2 x\} &= \mathscr{L}\{\frac{1+\cos(2x)}{2}\} = \frac{1}{2}\mathscr{L}\{1 + \cos(2x)\} \\ &= \frac{1}{2}\left(\mathscr{L}\{1\} + \mathscr{L}\{\cos(2x)\}\right) = \frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2+4}\right), \text{ for } s > 0.\end{aligned}$$