Inverse Laplace Transforms and Transforms of Derivatives

Inverse Laplace Transform

If $F(s) = \mathscr{L}\{f(x)\}$, then we say that $f(x)$ is the **inverse Laplace transform** of $F(s)$ and we write

$$f(x) = \mathscr{L}^{-1}\{F(s)\}.$$

In other words, the inverse Laplace transform of a function F is a function f whose Laplace transform equals to F. For example, from the previous lecture $\mathscr{L}\{1\} = \frac{1}{s}$. Hence, $\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1$. From the table of basic Laplace transforms, we can immediately obtain the table of basic inverse Laplace transforms:

$\mathscr{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = x^n, n = 1, 2, \ldots$	$\mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{ax}$
$\mathscr{L}^{-1}\left\{\frac{b}{s^2+b^2}\right\} = \sin(bx)$	$\mathscr{L}^{-1}\left\{\frac{s}{s^2+b^2}\right\} = \cos(bx)$
$\mathscr{L}^{-1}\left\{\frac{b}{s^2-b^2}\right\} = \sinh(bx)$	$\mathscr{L}^{-1}\left\{\frac{s}{s^2-b^2}\right\} = \cosh(bx)$

The inverse Laplace transform is also a linear operator, that is,

$$\mathscr{L}^{-1}\{c_1F(s) + c_2G(s)\} = c_1\mathscr{L}^{-1}\{F(s)\} + c_2\mathscr{L}^{-1}\{G(s)\}.$$

Example 1: Find Inverse Transforms

1. $\mathscr{L}^{-1}\left\{\frac{2}{s^3}\right\}$
2. $\mathscr{L}^{-1}\left\{\frac{1}{s^5}\right\}$
3. $\mathscr{L}^{-1}\left\{\frac{s}{s^2+4}\right\}$
4. $\mathscr{L}^{-1}\left\{\frac{1}{s^2+7}\right\}$

Solution

1. From the formula $\mathscr{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = x^n$, we have $\mathscr{L}^{-1}\left\{\frac{2}{s^3}\right\} = x^2$.
2. We need to match the form in the basic transform formula, so we need to rewrite the function $F(s)$. We have
$$\mathscr{L}^{-1}\left\{\frac{1}{s^5}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{4!}\cdot\frac{4!}{s^5}\right\} = \frac{1}{4!}\mathscr{L}^{-1}\left\{\frac{4!}{s^5}\right\} = \frac{1}{24}x^4.$$
where in the next to last step we have used linearity of the inverse Laplace transform.
3. From the formula $\mathscr{L}^{-1}\left\{\frac{s}{s^2+b^2}\right\} = \cos(bx)$, we have $\mathscr{L}^{-1}\left\{\frac{s}{s^2+4}\right\} = \cos(2x)$.
4. The formula $\mathscr{L}^{-1}\left\{\frac{b}{s^2+b^2}\right\} = \sin(bx)$ is applicable. We do need to rewrite the function $F(s)$ first. We have
$$\mathscr{L}^{-1}\left\{\frac{1}{s^2+7}\right\} = \frac{1}{\sqrt{7}}\mathscr{L}^{-1}\left\{\frac{\sqrt{7}}{s^2+7}\right\} = \frac{1}{\sqrt{7}}\sin(\sqrt{7}x).$$

Example 2: Termwise division and Linearity

Find $\mathscr{L}^{-1}\left\{\dfrac{3s+1}{s^2+2}\right\}$.

Solution

We start by splitting up the expression and applying linearity:

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{3s+1}{s^2+2}\right\} &= \mathscr{L}^{-1}\left\{\frac{3s}{s^2+2}+\frac{1}{s^2+2}\right\}\\ &= 3\mathscr{L}^{-1}\left\{\frac{s}{s^2+(\sqrt{2})^2}\right\}+\frac{1}{\sqrt{2}}\cdot\mathscr{L}^{-1}\left\{\frac{\sqrt{2}}{s^2+(\sqrt{2})^2}\right\}\\ &= 3\cos(\sqrt{2}x)+\frac{1}{\sqrt{2}}\sin(\sqrt{2}x).\end{aligned}$$

Example 3: Partial Fractions with Distinct Linear Factors

Find $\mathscr{L}^{-1}\{F\}$ where $F(s)=\dfrac{6s^2-13s+2}{s(s-1)(s-6)}$

Solution

We begin by finding the partial fraction decomposition of $F(s)$.

$$\frac{6s^2-13s+2}{s(s-1)(s-6)}=\frac{A}{s}+\frac{B}{s-1}+\frac{C}{s-6}.$$

Since the denominator is a product of distinct linear factors, we can use a shortcut called the "covered up" method to find A, B and C.

$$A=\frac{6s^2-13s+2}{(s-1)(s-6)}\Big|_{s=0}=\frac{1}{3},\quad B=\frac{6s^2-13s+2}{s(s-6)}\Big|_{s=1}=1,\quad C=\frac{6s^2-13s+2}{s(s-1)}\Big|_{s=6}=\frac{14}{3}.$$

We have

$$\mathscr{L}^{-1}\left\{\frac{6s^2-13s+2}{s(s-1)(s-6)}\right\}=\frac{1}{3}\mathscr{L}^{-1}\left\{\frac{1}{s}\right\}+\mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\}+\frac{14}{3}\mathscr{L}^{-1}\left\{\frac{1}{s-6}\right\}=\frac{1}{3}+e^x+\frac{14}{3}e^{6x}.$$

Transforms of Derivatives

Our goal is to use the Laplace transform to solve linear differential equations. The first step is to study the effect of the Laplace transform on the derivatives of a function. If $f(x)$ is a continuous function on $[0,\infty)$ and $\mathscr{L}\{f(x)\}=F(s)$, then

$$\mathscr{L}\{f'(x)\}=sF(s)-f(0).$$

More generally, we have

$$\mathscr{L}\{f^{(n)}(x)\}=s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)-\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0).$$

Roughly speaking, these important formulas show that we can use the Laplace transform to replace *differentiation with respect to x* with *multiplication by s*. Hence, it can be used to convert a differential equation into an algebraic equation.

When we solve a differential equation, the unknown function is often denoted by $y=y(x)$. Let $Y(s)=\mathscr{L}\{y\}$, the above formulas will become

$$\mathscr{L}\{y'\}=sY(s)-y(0).$$

$$\mathscr{L}\{y^{(n)}\}=s^nY(s)-s^{n-1}y(0)-s^{n-2}y'(0)-\cdots-sy^{(n-2)}(0)-y^{(n-1)}(0).$$

Example 4: Solve a First-Order IVP

Use the Laplace transform to solve the initial-value problem

1. $y' + 2y = 0$, $y(0) = 2$.
2. $y' - y = 2\cos(5x)$, $y(0) = 0$

Solution

1. We first apply the Laplace transform to both sides of the equation, and use linearity to obtain

$$\mathscr{L}\{y'\} + 2\mathscr{L}\{y\} = \mathscr{L}\{0\} = 0.$$

Using the property of Laplace transform of derivatives, we have

$$sY(s) - y(0) + 2Y(s) = 0,$$

where $Y(s) = \mathscr{L}\{y\}$. Substitute the initial condition $y(0) = 2$ into the equation and isolate $Y(s)$, we have

$$\begin{aligned} sY(s) - 2 + 2Y(s) &= 0 \\ (s+2)Y(s) &= 2 \\ Y(s) &= \frac{2}{s+2} \end{aligned}$$

Finally, to obtain the solution y, we find the inverse Laplace transform of $Y(s)$. We have

$$y = \mathscr{L}^{-1}\{Y(s)\} = \mathscr{L}^{-1}\left\{\frac{2}{s+2}\right\} = 2e^{-2x}.$$

Hence, the solution to the IVP is $y = 2e^{-2x}$.

2. Follow the same stategy, we have

$$\begin{aligned} \mathscr{L}\{y'\} - \mathscr{L}\{y\} &= 2\mathscr{L}\{\cos(5x)\} \quad \text{(Apply the Laplace transform to both sides)} \\ sY(s) - \underbrace{y(0)}_{0} - Y(s) &= \frac{2s}{s^2+25} \quad \text{(Apply the Laplace transform of derivatives and also find transform of } 2\cos(5x)\text{)} \\ (s-1)Y(s) &= \frac{2s}{s^2+25} \\ Y(s) &= \frac{2s}{(s-1)(s^2+25)} \quad \text{(Solve for } Y(s)\text{)} \end{aligned}$$

Now we need to find the inverse Laplace transform of $Y(s)$. We begin by finding the partial fractions decomposition of $Y(s)$:

$$Y(s) = \frac{2s}{(s-1)(s^2+25)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+25}.$$

Solve for the coefficients A, B and C, we have $A = 1/13$, $B = -1/13$, $C = 25/13$. It follows that

$$\begin{aligned} \mathscr{L}^{-1}\{Y(s)\} = \mathscr{L}^{-1}\left\{\frac{2s}{(s-1)(s^2+25)}\right\} &= \mathscr{L}^{-1}\left\{\frac{1}{13}\cdot\frac{1}{s-1} - \frac{1}{13}\cdot\frac{s}{s^2+25} + \frac{25}{13}\cdot\frac{1}{s^2+25}\right\} \\ &= \frac{1}{13}\mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{13}\mathscr{L}^{-1}\left\{\frac{s}{s^2+25}\right\} + \frac{25}{13}\cdot\frac{1}{5}\mathscr{L}^{-1}\left\{\frac{5}{s^2+25}\right\} \\ &= \frac{1}{13}e^x - \frac{1}{13}\cos(5x) + \frac{5}{13}\sin(5x). \end{aligned}$$

Hence, the solution to the IVP is

$$y = \mathscr{L}^{-1}\{Y(s)\} = \frac{1}{13}e^x - \frac{1}{13}\cos(5x) + \frac{5}{13}\sin(5x).$$

Solving IVPs by Method of Laplace Transform

The steps for solving an IVP using Laplace Transform are:

1. Take the Laplace transform of both sides of the equation.
2. Use the properties of the Laplace transform and the initial conditions to obtain an algebraic equation. Solve this equation for $Y(s) = \mathscr{L}\{y\}$.
3. Find the inverse Laplace transform $\mathscr{L}^{-1}\{Y(s)\}$. This gives the solution y to the IVP.

Example 5: Solve a Second-Order IVP

Use the Laplace transform to solve the initial-value problem

$$y'' + 3y' + 2y = 12e^{2x}, y(0) = 1, y'(0) = -1.$$

Solution

Apply the method of Laplace transform, we have

$$\begin{aligned}
\mathscr{L}\{y''\} + 3\mathscr{L}\{y'\} + 2\mathscr{L}\{y\} &= 12\mathscr{L}\{e^{2x}\} \quad \text{(Apply the Laplace transform to both sides)}\\
s^2Y(s) - s\underbrace{y(0)}_{1} - \underbrace{y'(0)}_{-1} + 3sY(s) - 3\underbrace{y(0)}_{1} + 2Y(s) &= \frac{12}{s-2} \quad \text{(Transform of derivatives and find transform of } 12e^{2x}\text{)}\\
\left(s^2+3s+2\right)Y(s) - s + 1 - 3 &= \frac{12}{s-2}\\
(s+1)(s+2)Y(s) &= \frac{12}{s-2} + s + 2 = \frac{s^2+8}{s-2}\\
Y(s) &= \frac{s^2+8}{(s-2)(s+1)(s+2)}
\end{aligned}$$

To find the inverse Laplace transform of $Y(s)$, we first use the "covered up" method to find the partial fraction decomposition of $Y(s)$.

$$Y(s) = \frac{s^2+8}{(s-2)(s+1)(s+2)} = \frac{A}{s-2} + \frac{B}{s+1} + \frac{C}{s+2}$$

where

$$A = \frac{s^2+8}{(s+1)(s+2)}\Big|_{s=2} = 1, \quad , B = \frac{s^2+8}{(s-2)(s+2)}\Big|_{s=-1} = -3, \quad , C = \frac{s^2+8}{(s-2)(s+1)}\Big|_{s=-2} = 3.$$

Hence,

$$\mathscr{L}^{-1}\{Y(s)\} = \mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\} - 3\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\} + 3\mathscr{L}^{-1}\left\{\frac{1}{s+2}\right\} = e^{2x} - 3e^{-x} + 3e^{-2x}.$$

Therefore, the solution of the IVP is

$$y = \mathscr{L}^{-1}\{Y(s)\} = e^{2x} - 3e^{-x} + 3e^{-2x}.$$