Operational Properties of the Laplace Transform - I

Translation in s

Translation in s property: If $\mathscr{L}\{f(x)\} = F(s)$ and a is any real number then,

$$\mathscr{L}\{e^{ax} f(x)\} = F(s-a).$$

This says that to find the Laplace transform of e^{ax} times a function, we just need to replace each of the s in the Laplace transform of that function by $s-a$. We can add the following formulas to the table of basic Laplace transforms:

Function $f(x)$	Laplace Transform $F(s) = \mathscr{L}\{f(x)\}$
$e^{ax}x^n$, $n = 1, 2, \ldots$	$\dfrac{n!}{(s-a)^{n+1}}$, $s > a$
$e^{ax}\sin(bx)$	$\dfrac{b}{(s-a)^2+b^2}$, $s > a$
$e^{ax}\cos(bx)$	$\dfrac{s-a}{(s-a)^2+b^2}$, $s > a$

Inverse Form of Translation in s property: If $f(x) = \mathscr{L}^{-1}\{F(s)\}$, then

$$\mathscr{L}^{-1}\{F(s-a)\} = e^{ax} f(x).$$

Example 1: Find Laplace transforms using the translation in s property

Find the Laplace transform

1. $\mathscr{L}\left\{e^{-2x}\cos(4x)\right\}$
2. $\mathscr{L}\left\{e^{3x}\left(9 - 4x + 10\sin\dfrac{x}{2}\right)\right\}$.

Solution

Write the solution here

Example 2: Find inverse Laplace transforms using the translation in s property

Find the inverse Laplace transforms:

1. $\mathscr{L}^{-1}\left\{\frac{1}{(s-1)^4}\right\}$
2. $\mathscr{L}^{-1}\left\{\frac{2s^2+10s}{(s^2-2s+5)(s+1)}\right\}$
3. $\mathscr{L}^{-1}\left\{\frac{5s}{(s-2)^2}\right\}$.

Solution

Write the solution here

Example 3: Solve an IVP by using the Laplace transform

1. $y''-4y'+4y=x^3e^{2x}$, $y(0)=y'(0)=0$
2. $y''-2y'+5y=-8e^{-x}$, $y(0)=2$, $y'(0)=12$.

Solution

Write the solution here

Unit Step Functions

The **unit step function** or the **Heaviside function** is defined as

$$\mathcal{U}(x-a)=\begin{cases}0 & 0\le x<a\\ 1 & x\ge a.\end{cases}$$

We can think about this function as being "off" on the interval $[0,a)$ and being "on" on the interval $[a,\infty)$. Also, note that

$$1-\mathcal{U}(x-a)=\begin{cases}1 & 0\le x<a\\ 0 & x\ge a.\end{cases}$$

So $1-\mathcal{U}(x-a)$ is "on" on $[0,a)$ and "off" on $[a,\infty)$. And for $0<a<b$

$$\mathcal{U}(x-a)-\mathcal{U}(x-b)=\begin{cases}0 & 0\le x<a\\ 1 & a\le x<b\\ 0 & x\ge b.\end{cases}$$

So, $\mathcal{U}(x-a)-\mathcal{U}(x-b)$ is "off" on $[0,a)$, "on" on $[a,b)$ and "off" on $[b,\infty)$.
We can use unit step functions to "turn on" and "turn off" any function f on prescribed intervals. As a result, any piecewise function can be written as a combination of unit step functions.
For example, the piecewise function

$$f(x)=\begin{cases}g(x) & 0\le x<a\\ h(x) & x\ge a\end{cases}$$

can be written as $f(x)=(1-\mathcal{U}(x-a))g(x)+\mathcal{U}(x-a)h(x)$. (We turn on g on $[0,a)$ and turn off g on $[a,\infty)$ by using $1-\mathcal{U}(x-a)$. Turn off h on $[0,a)$ and turn on h on $[a,\infty)$ by using $\mathcal{U}(x-a)$).
The piecewise function For example, the piecewise function

$$f(x)=\begin{cases}g(x) & 0\le x<a\\ h(x) & a\le x<b\\ j(x) & x\ge b\end{cases}$$

can be written as $f(x)=(1-\mathcal{U}(x-a))g(x)+(\mathcal{U}(x-a)-\mathcal{U}(x-b))h(x)+\mathcal{U}(x-b)j(x)$.
We will see the advantage of rewriting piecewise functions as combinations of unit step functions when we learn the second translation theorem.

Example 4: Write a piecewise function as a combination of unit step functions

Express the function as a combination of unit step functions.

$$f(x)=\begin{cases}3 & 0\le x<2\\ 1 & 2\le x<5\\ x & 5\le x<7\\ x^2 & x\ge 7.\end{cases}$$

Solution

Write the solution here

Translation in x

Laplace transform of the unit step function: The Laplace transform of $\mathcal{U}(x-a)$ with $a \geq 0$ is

$$\mathscr{L}\{\mathcal{U}(x-a)\} = \frac{e^{-as}}{s}.$$

Translation in x property: If $\mathscr{L}\{f(x)\} = F(s)$ and $a \geq 0$, then

$$\mathscr{L}\{f(x-a)\mathcal{U}(x-a)\} = e^{-as}F(s).$$

In practice, we often need to find $\mathscr{L}\{g(x)\mathcal{U}(x-a)\}$. We can rewrite the above property by identifying $g(x)$ with $f(x-a)$ (hence, $f(x) = g(x+a)$). Thus, an alternative (and useful) way to express the translation in x property is this:

Translation in x property - alternative form: If $\mathscr{L}\{g(x+a)\} = G(s)$ and $a \geq 0$, then

$$\mathscr{L}\{g(x)\mathcal{U}(x-a)\} = e^{-as}G(s).$$

This says that to find the Laplace transform of a function $g(x)$ times the unit step function $\mathcal{U}(x-a)$, we multiply the Laplace transform of $g(x+a)$ by e^{-ax}.

Inverse Form of Translation in x property: If $f(x) = \mathscr{L}^{-1}\{F(s)\}$, then

$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(x-a)\mathcal{U}(x-a).$$

This says that to find the inverse Laplace transform of a function of the form $e^{-as}F(s)$, we first find the inverse tranform of $F(s)$. Then we replace each x in the inverse transform of $F(s)$ by $x-a$ and multiply the resulting function by the unit step function $\mathcal{U}(x-a)$.

Example 5: Find Laplace transform using translation in x property

Find the Laplace transform:

1. $\mathscr{L}\left\{(x-1)^2\mathcal{U}(x-1)\right\}$
2. $\mathscr{L}\left\{x^2\mathcal{U}(x-1)\right\}$
3. $\mathscr{L}\left\{\sin(x)\mathcal{U}\left(x-\frac{\pi}{2}\right)\right\}$

Solution

Write the solution here

Example 6: Find Laplace transform of a piecewise function using translation in x property

Find the Laplace transform of the piecewise function in Example 4.

Solution

Write the solution here

Example 7: Find inverse Laplace transform using translation in x property

Find the inverse Laplace transform:

1. $\mathscr{L}^{-1}\left\{\dfrac{se^{-\pi s/2}}{s^2+4}\right\}$
2. $\mathscr{L}^{-1}\left\{\dfrac{(1+e^{-2s})^2}{s+2}\right\}$

Solution

Write the solution here

Example 8: Solve an IVP using Laplace transform

Solve the IVP:

$$y'' + 5y' + 6y = g(x), y(0) = 0, y'(0) = 2,$$

where

$$g(x) = \begin{cases} 0 & 0 \leq x < 1 \\ x & 1 \leq x < 5 \\ 1 & x \geq 5 \end{cases}$$

Solution

Write the solution here