Operational Properties of the Laplace Transform - II

Derivatives of the Laplace Transform

If $F(s) = \mathscr{L}\{f(x)\}$ and $n = 1, 2, 3\ldots,$ then

$$\mathscr{L}\{x^n f(x)\} = (-1)^n \frac{d^n F}{ds^n}(s).$$

This says that the Laplace transform of x^n times f equals $(-1)^n$ times the nth derivative of the Laplace transform of f.

Example 1: Find Laplace Transforms using Derivatives of the Laplace Transform

1. $\mathscr{L}\{x\sin(bx)\}$.
2. $\mathscr{L}\left\{x^2\cos(x)\right\}$
3. $\mathscr{L}\left\{xe^{-3x}\cos(3x)\right\}$

Solution

Write the solution here

Example 2: Solve an IVP

Solve the IVP: $y'' + 2xy' - 4y = 1$, $y(0) = y'(0) = 0$.
Note: we will need to use the fact that $\lim_{s\to\infty} Y(s) = 0$ where $Y(s) = \mathscr{L}\{y\}$.

Solution

Write the solution here

Laplace Transforms of Integrals

We have seen that the Laplace transform of the product of two functions is, in general, different from the product of the Laplace transforms of the individual functions, that is, in general $\mathscr{L}\{f \cdot g\} \neq \mathscr{L}\{f\} \cdot \mathscr{L}\{g\}$.
There is a special type of product that is "well-behaved" under the Laplace transform, it is called **convolution**.
Definition of the Convolution of Two Functions: Let f and g be continuous functions on $[0, \infty)$. The **convolution** of f and g, denoted by $f * g$, is the function defined as

$$(f * g)(x) = \int_0^x f(x-t)g(t)dt.$$

Example 3: Find the Convolution of Two Functions

Find the convolution $f * g$ of the functions $f(x) = e^x$ and $g(x) = \sin x$. Then find $\mathscr{L}\{(f * g)(x)\}$.

Solution

Write the solution here

Properties of Convolution

Basic Properties of Convolution: If f, g, and h are continuous functions on $[0, \infty)$, then

1. $f * g = g * f$
2. $f * (g + h) = (f * g) + (f * h)$
3. $(f * g) * h = f * (g * h)$
4. $f * 0 = 0$

Convolution Theorem: If f and g are continuous functions on $[0, \infty)$ and $F(s) = \mathscr{L}\{f(x)\}$ and $G(s) = \mathscr{L}\{g(x)\}$, then

$$\mathscr{L}\{(f * g)(x)\} = \mathscr{L}\{f(x)\}\mathscr{L}\{g(x)\} = F(s)G(s).$$

This says that the Laplace transform of the convolution of two functions equals the product of the Laplace transforms of the functions.

Inverse Form of the Convolution Theorem: If $F(s) = \mathscr{L}\{f(x)\}$ and $G(s) = \mathscr{L}\{g(x)\}$, then

$$\mathscr{L}^{-1}\{F(s)G(s)\} = (f * g)(x).$$

Example 4: Find Inverse Laplace Transforms Using the Convolution Theorem

Use the convolution theorem to find the inverse Laplace transform $\mathscr{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}$

Solution

Write the solution here

Example 5: Solve Integral Equations

Use the Laplace transform to solve the integral equation:

$$y(x) + \int_0^x e^{x-t} y(t)\,dt = \sin x.$$

Solution

Write the solution here