Preliminary Theorey - Linear Systems

Linear Systems of Differential Equations

Let x and y be functions of a (time) variable t. A **linear system of 2 first order equations** is a system of the form

$$\frac{dx}{dt} = a_1(t)x + b_1(t)y + f_1(t)$$
$$\frac{dy}{dt} = a_2(t)x + b_2(t)y + f_2(t)$$

where the functions $a_j(t), b_j(t), f_j(t)$, $j = 1, 2$, are continuous on a common interval I. If $f_j(t) = 0$ for $j = 1, 2$, the system is a **homogeneous system**; otherwise, it is a **nonhomogeneous system**.
A **solution** to the system is a set of functions $x(t)$, $y(t)$ defined on I that satisfies both equations in the system.
If we also require that $x(t)$ and $y(t)$ satisfy the initial conditions: $x(t_0) = \gamma_1$ and $y(t_0) = \gamma_2$ for some constants γ_1 and γ_2, then we have an initial value problem (IVP).
More generally, let $x_1, \ldots, x_n$ be functions of t. A **linear system of n first order equations** has the form

$$\frac{dx_1}{dt} = a_{11}(t)x_1(t) + a_{12}(t)x_2(t) + \cdots + a_{1n}x_n(t) + f_1(t)$$
$$\frac{dx_2}{dt} = a_{21}(t)x_1(t) + a_{22}(t)x_2(t) + \cdots + a_{2n}x_n(t) + f_2(t)$$
$$\vdots$$
$$\frac{dx_n}{dt} = a_{n1}(t)x_1(t) + a_{n2}(t)x_2(t) + \cdots + a_{nn}x_n(t) + f_n(t)$$

Example 1: Verification of Solutions

Verify that the set of functions $x(t) = 5e^t \cos t$ and $y(t) = 3e^t \cos t - e^t \sin t$ is a solution to the system

$$\frac{dx}{dt} = -2x + 5y$$
$$\frac{dy}{dt} = -2x + 4y$$

Solution

Write the solution here

Transform an nth-order equations to a system

Any nth-order differential equations can be transformed into a system of first-order differential equations. As an example, consider the 3rd-order equation in the function $y = y(t)$

$$y''' - 2y'' + 3y' - 5y = 7t^2.$$

We re-label y and the sequence of derivatives of y as

$$x_1(t) := y(t), x_2(t) := y'(t), x_3(t) := y''(t).$$

Note that $x_3' = y'''$. Substitute this into the original equation and isolate x_3', we obtain

$$x_3' = 2x_3 - 3x_2 + 5x_1 + 7t^2.$$

This together with the defining equations for x_2 and x_3 gives the system

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= 2x_3 - 3x_2 + 5x_1 + 7t^2 \end{aligned}$$

In general an nth-order differential equation of the form

$$y^{(n)}(t) = f(t, y, y', y'', \ldots y^{(n-1)})$$

can always be transformed to a system of m first order equations by the substitution:

$$x_1(t) := y(t), x_2(t) := y'(t), \ldots, x_n(t) := y^{(n-1)}(t).$$

The last equation in the system will be

$$x_n' = f(t, x_1, x_2, x_3, \ldots, x_n).$$

Example 2: Convert an nth-order equation into a system

Convert the equation

$$y'' + ty' - 3y = t^2$$

into a system of first order equations.

Solution

Write the solution here

The Method of Eigenvalues

Consider the **homogeneous linear system of 2 first order equations with constant coefficients**

$$x' = a_1x + b_1y$$
$$y' = a_2x + b_2y$$

Here a_1, b_1, a_2, b_2 are real constants. The **characteristic equation** of this linear system is the equation

$$\lambda^2 - (a_1 + b_2)\lambda + a_1b_2 - a_2b_1 = 0.$$

The roots of this equation are called the **eigenvalues** of the system. There are 3 possible scenarios when we solve this equation:

1. The roots λ_1 and λ_2 are real and distinct: in this case, we obtain
$$x(t) = C_1e^{\lambda_1 t} + C_2e^{\lambda_2 t}.$$
2. The roots λ_1 and λ_2 are real and repeated, i.e., $\lambda_1 = \lambda_2$: in this case, we have
$$x(t) = C_1e^{\lambda_1 t} + C_2te^{\lambda_1 t}.$$
3. The roots λ_1 and λ_2 are complex (non-real), i.e., $\lambda_{1,2} = \alpha \pm i\beta$: in this case, we have
$$x(t) = e^{\alpha t}(C_1 \sin(\beta t) + C_2 \cos(\beta t)).$$

   Once, we obtain $x(t)$, we can substitute $x(t)$ and $x'(t)$ into the first equation and solve for the remaining unknown function $y(t)$.

Example 3: Solve a homogeneous linear system of 2 first order equations

Solve the IVP

$$x' = 4x + 6y$$
$$y' = -3x - 5y$$

$x(0) = 1$, $y(0) = 0$.

Solution

Write the solution here