Direction fields

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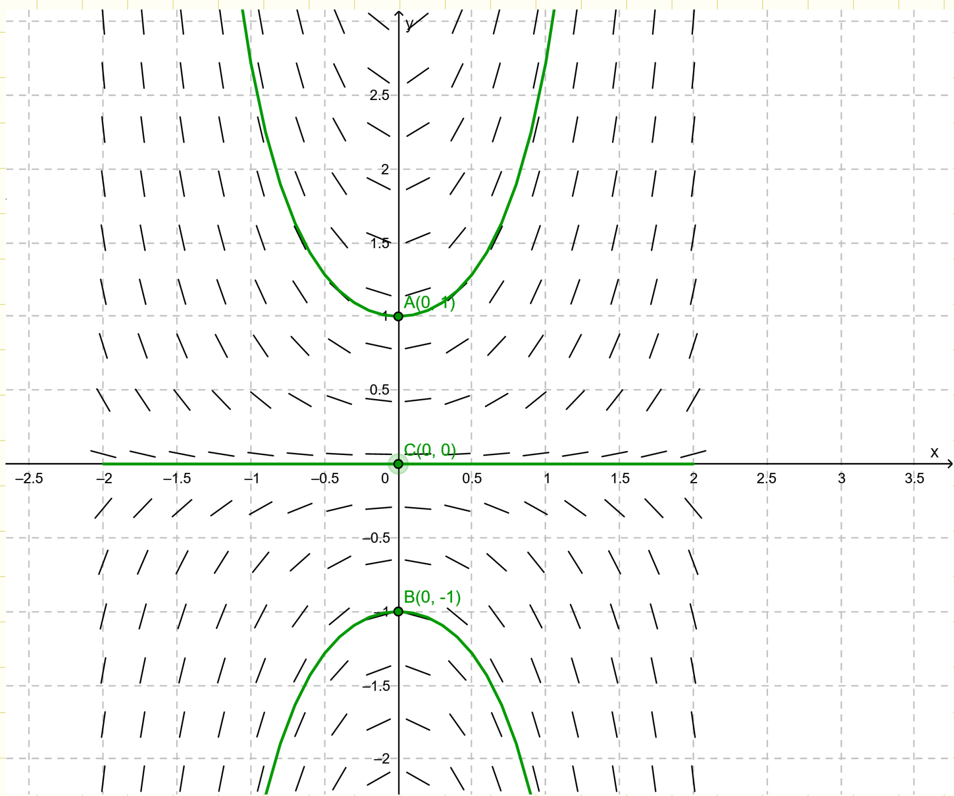
Consider the first order ODE $\dfrac{dy}{dx} = f(x, y)$. The quantity $\frac{dy}{dx}$ at a point (x, y) is the slope of the tangent line at (x, y) to the solution function y. Hence, the value of the function $f(x, y)$ at any point (x, y) is the slope of y at that point. If we systematically evaluate f at various points (x, y) in the plane and plot short line segments at each point (x, y) with slope $f(x, y)$, then we obtain the **direction field** or the **slope field** of the equation $\dfrac{dy}{dx} = f(x, y)$. The direction field suggests the shape of a family of solution curves of the differential equation.

Example 1: Direction field

Consider the first order ODE $\dfrac{dy}{dx} = 2xy$. Sketch the direction field. Sketch the solution curves that pass through $(0, 1)$, $(0, -1)$ and $(0, 0)$.

Solution



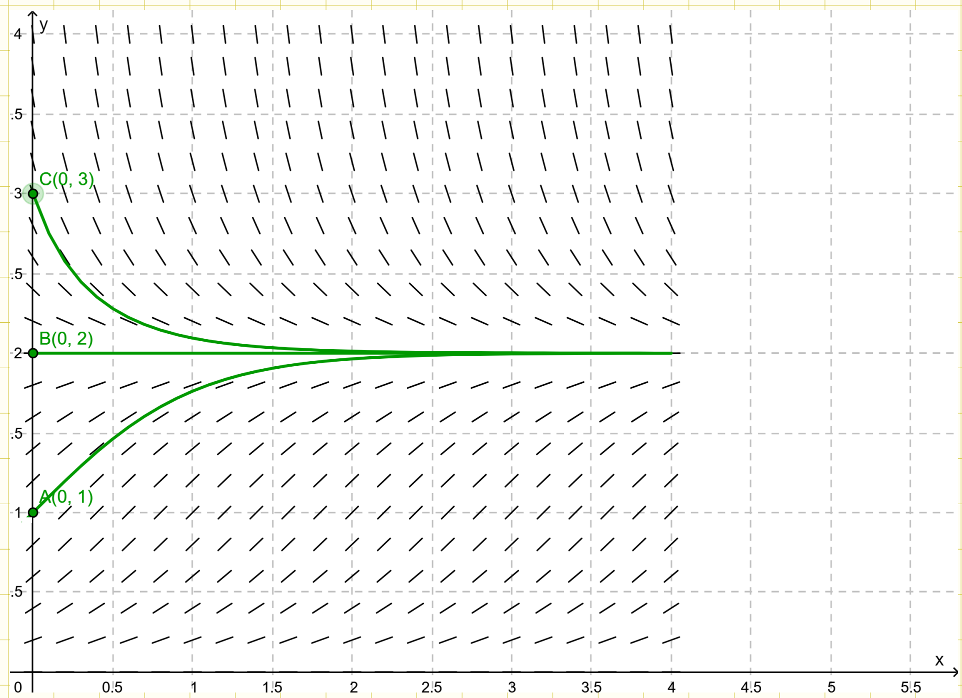
Example 2: Autonomous Equation

Consider the first order ODE $\dfrac{dy}{dt} = y(2-y)$ where $y(t)$ represent the population (in thousands) at time t of a certain species.

1. Sketch the direction field. Find the **equilibrium solutions**.
2. If the initial population is 1000, i.e., $y(0) = 1000$, what can you say about $\lim_{t\to\infty} y(t)$?
3. If the initial population is 3000, i.e., $y(0) = 3000$, what can you say about $\lim_{t\to\infty} y(t)$?

Solution

1. This is an example of an autonomous ODE: $\dfrac{dy}{dx} = f(y)$. The **equilibrium solutions** are the solutions $y = c$ where c is a zero of $f(y)$. For this particular equation the equilibrium solutions are $y = 0$ and $y = 2$.



2. From the direction field, we see that if $y(0) = 1000$, then $\lim_{t\to\infty} y(t) = 2000$. More generally, population that starts below 2000 will steadily increase and approach 2000.
3. From the direction field, we see that if $y(0) = 3000$, then $\lim_{t\to\infty} y(t) = 2000$. More generally, population that starts above 2000 will steadily decrease and approach 2000.