Special types of first order equations - Linear Equations

Recommended reading from Zill's DEs with BVP-7e: Section 2.3 (pg 53-60): Examples 1 through 6.

Solve first order linear equations

A first order linear differential equation is an equation of the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

Method of solution of a linear first order equation:

1. Rewrite the equation in **standard form** by dividing both sides by $a_1(x)$ to obtain

$$\frac{dy}{dx} + P(x)y = Q(x).$$

2. Find the **integrating factor** $I(x) = e^{\int P(x)dx}$.
3. Multiply both sides of the equation in Step 1 by $I(x)$ to obtain

$$I(x)\frac{dy}{dx} + I(x)P(x)y = I(x)Q(x).$$

4. By the product rule, the left hand side of the above equation becomes $\frac{d}{dx}[I(x)y]$, i.e., the derivative with respect to x of the product of the integrating factor and y. So, the above equation becomes

$$\frac{d}{dx}[I(x)y] = I(x)Q(x).$$

5. Integrate both sides of the above equation with respect to x to obtain

$$I(x)y = \int I(x)Q(x)dx + C$$

6. Isolate y by dividing both sides of the above equation by $I(x)$ to obtain a 1-parameter family of solutions.

Note: applying the formula $\frac{d}{dx}[e^u] = e^u\frac{du}{dx}$ with $u = \int P(x)dx$ we have

$$\frac{d}{dx}[I(x)] = \frac{d}{dx}\left[e^{\int P(x)dx}\right] = P(x)e^{\int P(x)dx} = P(x)I(x).$$

Applying the product rule $\frac{d}{dx}[uv] = u\frac{dv}{dx} + v\frac{du}{dx}$, we have

$$\frac{d}{dx}[I(x)y] = I(x)\frac{dy}{dx} + y\frac{d}{dx}[I(x)] = I(x)\frac{dy}{dx} + yP(x)I(x) = \text{left side of equation in Step 3.}$$

This is the justification for the claim at the beginning of Step 4.
In Step 5, when we integrate with respect to x the quantity $\frac{d}{dx}[I(x)y]$ on the left side, the $\frac{d}{dx}$ "goes away." This is because integration and differentiation are inverse operations of each other.
Note: It turns out that if $P(x)$ and $Q(x)$ are continuous on a common interval (a, b), then the 1-parameter family of solutions obtained by this method is the **true general solution** of the differential equation.

Example 1: Solve a linear first order equation

Find the general solution of the equation and determine the largest interval over which the solution is defined.

$$x\frac{dy}{dx} + 2y = x^{-3}.$$

Solution

Write the solution here

Example 2: Solve an IVP

Solve the initial value problem $y' + (\tan x)y = \cos^2 x, \qquad y(0) = -1.$

Solution

Write the solution here

Example 3: Discontinuous Forcing Term

In this example we solve the linear equation $\frac{dy}{dx} + P(x)y = Q(x)$ where the **forcing term** $Q(x)$ is a function with a jump discontinuity.

Consider the equation $\frac{dy}{dx} + y = Q(x)$, $y(0) = 1$ where

$$Q(x) = \begin{cases} 1, \text{ if } 0 \leq x \leq 1 \\ -1, \text{ if } x > 1 \end{cases}$$

1. Find the general solution for $0 \leq x \leq 1$ and choose the constant in the solution so that the initial condition is satisfied.

2. Find the general solution for $x > 1$ and choose the constant in the solution so that the solution from the previous part and the solution in this part agree at $x = 1$.

Note: By "glueing" the two solutions on the two intervals together, we obtain a continuous function that satisfies the equation except at $x = 1$ where its derivative in undefined.

Solution

Write the solution here