Special types of first order equations - Exact Equations

Recommended reading from Zill's DEs with BVP-7e: Section 2.4 (pg 62-68): Examples 1 through 4.

Exact Equations and Method for Solving Exact Equations

Let $F(x, y)$ be a function of 2 variables x and y. The **total differential** of F, denoted by dF is

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy,$$

where $\frac{\partial F}{\partial x}$ is the partial derivative of F with respect to x and $\frac{\partial F}{\partial y}$ is the partial derivative of F with respect to y.
A differential expression $M(x, y)dx + N(x, y)dy$ is said to be **exact** if there is a function $F(x, y)$ such that

$$\frac{\partial F}{\partial x}(x, y) = M(x, y) \text{ and } \frac{\partial F}{\partial y}(x, y) = N(x, y).$$

In other words, it is the total differential of $F(x, y)$:

$$dF(x, y) = M(x, y)dx + N(x, y)dy.$$

If $M(x, y)dx + N(x, y)dy$ is an exact differential expression, then the equation

$$M(x, y)dx + N(x, y)dy = 0$$

is called an **exact differential equation**. It follows that a 1-parameter family of solutions of the exact differential equation is

$$F(x, y) = C.$$

Test for Exactness: The differential equation $M(x, y)dx + N(x, y)dy = 0$ is an exact equation if and only if

$$\frac{\partial M}{\partial y}(x, y) = \frac{\partial N}{\partial x}(x, y),$$

here we assume $M(x, y)$, $N(x, y)$ are continuous and have continuous first partial derivatives in a rectangular region $R : a < x < b, c < y < d$.
Once we have confirmed that the equation $Mdx + Ndy = 0$ is exact, to solve it we need to find a function $F(x, y)$ such that $\frac{\partial F}{\partial x} = M$ and $\frac{\partial F}{\partial y} = N$. This suggests the following solution method.
Method for Solving Exact Equations:

1. Integrate M with respect to x, the "constant of integration" is a function of y, call it $g(y)$. So, we get

$$F(x, y) = \int M(x, y)dx + g(y)$$

   and we need to find $g(y)$.

2. Take the partial derivative with respect to y of both sides of the above equation and substitute N for the resulting left hand side $\frac{\partial F}{\partial y}$. We can then solve for $g'(y)$.

3. Integrate $g'(y)$ with respect to y to obtain $g(y)$, we can either add a C for the constant of integration at this step or omit adding a constant.

4. Substitute $g(y)$ into the right hand side of the equation in Step 1 to get $F(x, y)$.

5. The solution to the exact equation $Mdx + Ndy = 0$ is the 1-parameter family of solutions $F(x, y) = C$.

Example 1: Construct exact equations

Find the total differential dF of the given function $F(x,y)$. Use dF to construct an exact differential equation $Mdx + Ndy = 0$ whose solution is $F(x,y) = C$. Verify that $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

1. $F(x,y) = 3x^2y + 5xy + y^3 + 5$
2. $F(x,y) = x^2y - \tan x + y^2$

Solution

Write the solution here

Example 2: Solving an exact equation

Verify that the equation is exact and find a 1-parameter family of solutions of the equation

$$(3x^2y + 8xy^2)dx + (x^3 + 8x^2y + 12y^2)dy = 0.$$

Solution

Write the solution here

Example 3: Solving an exact equation

Verify that the equation is exact and solve the initial value problem

$$(y^2 \cos x - 3x^2 y - 2x)dx + (2y \sin x - x^3 + \ln y)dy = 0, \qquad y(0) = e.$$

Solution

Write the solution here

Special Integrating Factors

Sometimes we can solve a "nonexact" equation $Mdx + Ndy = 0$ by finding an **integrating factor** $I(x, y)$ so that after multiplying by $I(x, y)$ the equation

$$I(x, y)M(x, y)dx + I(x, y)N(x, y)dy = 0$$

is exact.

Method for finding integrating factors: Given a "nonexact" equation $Mdx + Ndy = 0$:

- If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ is a function of x alone, then an integrating factor is

$$I(x) = exp\left[\int \left(\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}\right) dx\right].$$

- If $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$ is a function of y alone, then an integrating factor is

$$I(y) = exp\left[\int \left(\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}\right) dy\right].$$

Example 4: Nonexact equations made exact

Show that the differential equation is not exact. Find an integrating factor to make it exact.

1. $(x^2 + y^2 + x)dx + xydy = 0$
2. $\cos x dx + \left(1 + \dfrac{2}{y}\right) \sin x dy = 0$

Solution

Write the solution here