# Solution by Substitutions

## First Order Homogeneous Equations

The first order equation

$$\frac{dy}{dx} = f(x, y)$$

is **homogeneous** if $f(x, y)$ can be expressed as a function of the ratio $y/x$ alone.

**Method for solving first order homogeneous equations:**

1. Transform the equation to the form $\frac{dy}{dx} = f(x, y)$ if it is not in that form.

2. Make the substitution $u = \frac{y}{x}$. Then $y = ux$. Hence, $\frac{dy}{dx} = u + x\frac{du}{dx}$.

3. Substitute the expression for $\frac{dy}{dx}$ into the original equation. We will obtain the equation

$$u + x\frac{du}{dx} = G(u).$$

   (Note that we have transform the right hand side into a function of $u = \frac{y}{x}$. This can be done because of the "homogeneous" assumption.)

4. The above equation is separable because it can be rewritten as

$$\frac{1}{G(u) - u}du = \frac{1}{x}dx.$$

5. Integrate both sides to obtain an implicit solution.

6. Express the solution in terms of the original variables $x$ and $y$.

## Example 1: Solve a homogeneous differential equation

Explain why the given equation is NOT separable, exact or linear. Rewrite the equation in the form $\frac{dy}{dx} = f(x, y)$. Explain why the equation is homogeneous and use the substitution method to solve.

$$(y^2 + yx)dx + x^2dy = 0.$$

## Solution

Write the solution here

**Solution**

Solution to Example 1 (Cont.)

**Example 2: Solve a homogeneous differential equation**

Explain why the given equation is NOT separable, exact or linear. Explain why the equation is homogeneous and use the substitution method to solve.

$$\frac{dy}{dx} = \frac{x+3y}{3x+y}.$$

**Solution**

Write the solution here

## Bernoulli Equation

The first order differential equation

$$\frac{dy}{dx} + P(x)y = Q(x)y^n,$$

where $n$ is any real number is called **Bernoulli's equation**. Note that if $n = 0$ or $n = 1$, the equation is linear. On the other hand, for $n \neq 0, 1$, this equation is nonlinear. The technique for solving a Bernoulli's equation when $n \neq 0, 1$ is as follows:

**Method for solving a nonlinear Bernoulli's equation:**

1. Divide both sides of the equation by $y^n$, we obtain

$$y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x).$$

2. Let $u = y^{1-n}$. By the chain rule, we have

$$\frac{du}{dx} = (1-n)y^{-n}\frac{dy}{dx}.$$

3. The equation in Step 1 then becomes

$$\frac{1}{1-n}\frac{du}{dx} + P(x)u = Q(x).$$

4. Since $\dfrac{1}{1-n}$ is a constant, the above equation is a first order linear equation. Hence, it can be solved using the technique in Lecture 4.

### Example 3: Solve a Bernoulli's equation

Solve the given differential equation:

$$\frac{dy}{dx} - y = e^{2x}y^3.$$

### Solution

Write the solution here

### Example 4: Solve an IVP

Solve the given IVP:

$$x^2 \frac{dy}{dx} - 2xy = 3y^4, y(1) = \frac{1}{2}.$$

### Solution

Write the solution here