Preliminary Theory - Linear Equations of order n

Initial Value Problems - Existence and Uniqueness Theorem

Recall that a linear differential equation of order n is an equation of the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

An nth-order initial value problem is the problem of solving the above equation subject to the conditions:

$$y(x_0) = y_0, y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1},$$

where x_0, y_0, $y_1, \ldots, y_{n-1}$ are constants.
Existence and Uniqueness Theorem for an nth-order IVP:
If $a_0(x)$, $a_1(x), \ldots, a_n(x)$ and $g(x)$ are continuous functions on an interval I containing the point x_0 and $a_n(x) \neq 0$ for every x in I, then the IVP has one and only one solution $y = y(x)$ on I.

Example 1: nth-order initial value problem

Consider the initial value problem

$$\text{Solve: } x(x-1)\frac{d^3y}{dx^3} - 3x\frac{d^2y}{dx^2} + 6x^2\frac{dy}{dx} - (\cos x)y = \sqrt{x+5}$$
$$\text{Subject to: } y(x_0) = 2, y'(x_0) = 6, y''(x_0) = 7.$$

Determine the largest interval $I = (a, b)$ for which the Existence and Uniqueness Theorem guarantees the existence of a unique solution on (a, b) if

1. $x_0 = -1$
2. $x_0 = 1/3$
3. $x_0 = 10$.

Solution

Write the solution here

Example 2: Find the unique solution

Consider the initial value problem

$$\text{Solve: } xy''' - y'' = -2$$
$$\text{Subject to: } y(1) = 2, y'(1) = -1, y''(1) = -4.$$

1. Determine the largest interval $I = (a, b)$ for which the IVP has a unique solution.
2. Given that the 3-parameter family of functions $y = c_1 + c_2x + c_3x^3 + x^2$ is the general solution of the equation $xy''' - y'' = -2$. Find a member of the family that is the unique solution of the IVP.
3. Suppose we replace the initial conditions by $y(0) = 0$, $y'(0) = 1$, $y''(0) = 3$. Show that constants c_1, c_2 and c_3 cannot be found so that a member of the family satisfies these conditions. Explain why this does not violate the Existence and Uniqueness Theorem.

Solution

Write the solution here

Linear Dependence and Linear Independence

A collection of n functions $f_1(x)$, $f_2(x), \ldots, f_n(x)$ is said to be **linearly dependent** on an interval I if we can find constants c_1, $c_2, \ldots, c_n$ **not all zero**, such that

$$c_1f_1(x) + c_2f_2(x) + \ldots + c_nf_n(x) = 0$$

for every x in the interval I.
A collection of functions is **linearly independent** if it is NOT linearly dependent.
It follows directly from the definition that a collection of two functions $f_1(x)$ and $f_2(x)$ are linearly dependent if one is a constant multiple of the other.

Example 3: Linearly dependent/independent

Determine whether the given collection of functions is linearly dependent or linearly independent on the interval. Justify the answer.

1. $\{x^2, x^2 - 1, 5\}$ on $I = (-\infty, \infty)$
2. $\{\sin x, \cos x, \tan x\}$ on $I = (-\pi/2, \pi/2)$

Solution

Write the solution here

Homogeneous Equations

A linear differential equation of order n of the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

is said to be **homogeneous**.

Theorem - Solutions of Homogeneous Linear Equations

1. If $a_0(x)$, $a_1(x), \ldots, a_n(x)$ and $g(x)$ are continuous functions on an interval I and $a_n(x) \neq 0$ for every x in I, then the homogeneous linear equation of order n has n linearly independent solutions $y_1(x)$, $y_2(x), \ldots, y_n(x)$ on I. This collection of functions is called a **fundamental set of solutions** for the equation on I.

2. The **linear combination** of these n solutions

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x)$$

where c_1, $c_2, \ldots, c_n$ are arbitrary constants is also a solution. The linear combination of $y_1, \ldots, y_n$ written with arbitrary constants $c_1, \ldots, c_n$ is referred to as the **general solution** to the homogeneous linear equation.

Example 4: Fundamental set of solutions for a homogeneous equation

Verify that the given collection of functions form a fundamental set of solutions for the equation on the indicated interval. Form the general solution to the equation.

$$4y'' - 4y' + y = 0, \{e^{x/2}, xe^{x/2}\}, (-\infty, \infty).$$

Solution

Write the solution here

Nonhomogeneous Equations

A linear differential equation of order n of the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

where $g(x)$ is NOT identically zero, i.e., g is NOT the zero function is said to be **nonhomogeneous**.

Theorem - Solutions of Nonhomogeneous Linear Equations

Suppose that the same conditions about the coefficient functions $a_0(x), \ldots, a_n(x)$ as in the homogeneous case are satisfied. If $y_p(x)$ is a **particular solution of the nonhomogeneous equation** on the interval I and $y_1(x), \ldots, y_n(x)$ form a fundamental set of solutions for the associated homogeneous equation on I, then every solution of the nonhomogeneous equation on I can be expressed in the form

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x) + y_p(x)$$

where the $c_1, \ldots, c_n$ are arbitrary constants.

The linear combination of $y_p, y_1, \ldots, y_n$ written with arbitrary constants $c_1, \ldots, c_n$ is referred to as the general solution to the nonhomogeneous equation on I.

Note: we call the combination

$$y_c(x) = c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x)$$

the **complementary function** of the nonhomogeneous equation. In other words, to solve a nonhomogeneous linear equation, we first solve the associated homogeneous equation to obtain the complementary function. Then we try to find a particular solution of the nonhomogeneous equation. The general solution to the nonhomogeneous equation is then expressed as

$$y = \text{ complementary function } + \text{ any particular solution } = y_c + y_p.$$

Example 5: General solution to a nonhomogeneous equation

Verify that $y_p(x) = -3\sin x + 4\cos x$ is a particular solution of the nonhomogeneous equation

$$4y'' - 4y' + y = 25\sin x.$$

In Example 4, we are given a fundamental set of solutions for the associated homogeneous equation. Apply the theorem above to find the general solution to the nonhomogeneous equation.

Solution

Write the solution here

Differential Operator

The symbol D is called a **differential operator**. In particular, if $y = y(x)$, then $Dy = \frac{dy}{dx}$, that is, D transforms a function to its first derivative function. For example, $D(\cos x) = -\sin x$, $D(x^3 + 2x^2 - 7x) = 3x^2 + 4x - 7$. Similarly, D^2 transforms a function to its second derivative function. For example, $D^2(x^3) = 6x$. In general,

$$D^n y = \frac{d^n y}{dx^n} \text{(the nth derivative of } y\text{)}.$$

Polynomial expressions that involve D are also differential operators. For example, $(D^2 - 3D + 2)y$ equals to $D^2y - 3Dy + 2y$. Similarly, $(x^3D^3 - 2x^2D^2 + 3xD + 7)y = x^3D^3y - 2x^2D^2y + 3xDy + 7y$. In general, we define an nth-order linear differential operator to be

$$L = a_n(x)D^n + a_{n-1}(x)D^{n-1} + \ldots + a_1(x)D + a_0(x).$$

So, the generic form of a linear differential equation of order n can be rewritten as

$$Ly = g(x).$$

Example 6: Working with differential operators

Let $L_1 = D^2 + D - 2$ and $L_2 = x^3D^3 + x^2D^2 - xD + 6$. Calculate L_1y and L_2y where $y = x^3 - 1$.

Solution

Write the solution here

Example 7: Factor a differential operator

Show that the operator $D^2 + 3D + 2$ is the same as the operator $(D + 2)(D + 1)$, that is, show that for any differentiable function y, we have

$$(D^2 + 3D + 2)y = (D + 2)(D + 1)y.$$

Solution

Write the solution here