Initial Value Problems - Existence and Uniqueness Theorem

Recall that a linear differential equation of order n is an equation of the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

An nth-order initial value problem is the problem of solving the above equation subject to the conditions:

$$y(x_0) = y_0, y'(x_0) = y_1, \dots, y^{(n-1)}(x_0) = y_{n-1},$$

where $x_0, y_0, y_1, \ldots, y_{n-1}$ are constants.

Existence and Uniqueness Theorem for an nth-order IVP:

If $a_0(x)$, $a_1(x)$, ..., $a_n(x)$ and g(x) are continuous functions on an interval I containing the point x_0 and $a_n(x) \neq 0$ for every x in I, then the IVP has one and only one solution y = y(x) on I.

Example 1: nth-order initial value problem Consider the initial value problem Solve: $x(x-1)\frac{d^3y}{dx^3} - 3x\frac{d^2y}{dx^2} + 6x^2\frac{dy}{dx} - (\cos x)y = \sqrt{x+5}$

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Linear Dependence and Linear Independence

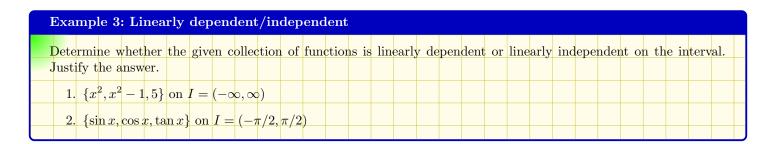
A collection of n functions $f_1(x), f_2(x), \ldots, f_n(x)$ is said to be **linearly dependent** on an interval I if we can find constants c_1, c_2, \ldots, c_n not all zero, such that

$$c_1 f_1(x) + c_2 f_2(x) + \ldots + c_n f_n(x) = 0$$

for every x in the interval I.

A collection of functions is **linearly independent** if it is NOT linearly dependent.

It follows directly from the definition that a collection of two functions $f_1(x)$ and $f_2(x)$ are linearly dependent if one is a constant multiple of the other.



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Homogeneous Equations

A linear differential equation of order n of the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

is said to be homogeneous. Theorem - Solutions of Homogeneous Linear Equations

- 1. If $a_0(x)$, $a_1(x)$, ..., $a_n(x)$ and g(x) are continuous functions on an interval I and $a_n(x) \neq 0$ for every x in I, then the homogeneous linear equation of order n has n linearly independent solutions $y_1(x)$, $y_2(x)$, ..., $y_n(x)$ on I. This collection of functions is called a **fundamental set of solutions** for the equation on I.
- 2. The linear combination of these n solutions

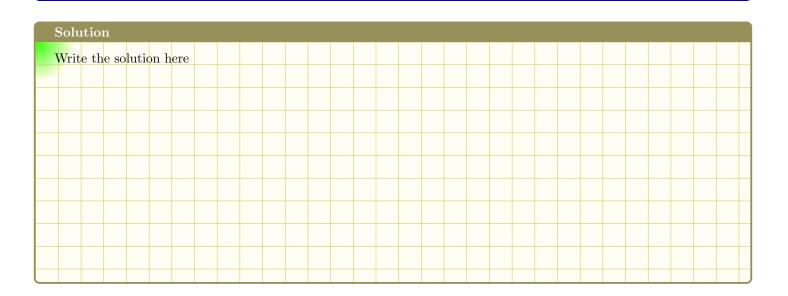
$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x)$$

where c_1, c_2, \ldots, c_n are arbitrary constants is also a solution. The linear combination of y_1, \ldots, y_n written with arbitrary constants c_1, \ldots, c_n is referred to as the **general solution** to the homogeneous linear equation.

Example 4: Fundamental set of solutions for a homogeneous equation

 Verify that the given collection of functions form a fundamental set of solutions for the equation on the indicated interval. Form the general solution to the equation.

 $4y'' - 4y' + y = 0, \{e^{x/2}, xe^{x/2}\}, (-\infty, \infty).$



A linear differential equation of order n of the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

where g(x) is NOT identically zero, i.e., g is NOT the zero function is said to be **nonhomogeneous**. Theorem - Solutions of Nonhomogeneous Linear Equations

Suppose that the same conditions about the coefficient functions $a_0(x), \ldots, a_n(x)$ as in the homogeneous case are satisfied. If $y_p(x)$ is a **particular solution of the nonhomogeneous equation** on the interval I and $y_1(x), \ldots, y_n(x)$ form a fundamental set of solutions for the associated homogeneous equation on I, then every solution of the nonhomogeneous equation on I can be expressed in the form

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x) + y_p(x)$$

where the c_1, \ldots, c_n are arbitrary constants.

The linear combination of y_p, y_1, \ldots, y_n written with arbitrary constants c_1, \ldots, c_n is referred to as the general solution to the nonhomogeneous equation on I.

Note: we call the combination

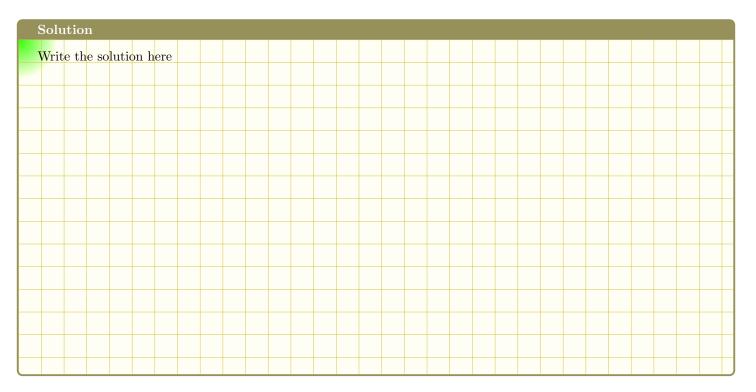
$$y_c(x) = c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x)$$

the **complementary function** of the nonhomogeneous equation. In other words, to solve a nonhomogeneous linear equation, we first solve the associated homogeneous equation to obtain the complementary function. Then we try to find a particular solution of the nonhomogeneous equation. The general solution to the nonhomogeneous equation is then expressed as

y = complementary function + any particular solution = $y_c + y_p$.

Example 5: General solution to a nonhomogeneous equation

Verif	y th	iat į	$J_p(x)$) =	-38	$\sin x$	+4	\cos	x is	a pa	artic	ulaı	r sol	utio	n of	the	nor	nhon	noge	neou	ıs e	quat	ion							
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The symbol D is called a **differential operator**. In particular, if y = y(x), then $Dy = \frac{dy}{dx}$, that is, D transforms a function to its first derivative function. For example, $D(\cos x) = -\sin x$, $D(x^3 + 2x^2 - 7x) = 3x^2 + 4x - 7$. Similarly, D^2 transforms a function to its second derivative function. For example, $D^2(x^3) = 6x$. In general,

$$D^n y = \frac{d^n y}{dx^n}$$
 (the nth derivative of y).

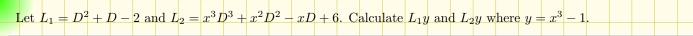
Polynomial expressions that involve D are also differential operators. For example, $(D^2 - 3D + 2)y$ equals to $D^2y - 3Dy + 2y$. Similarly, $(x^3D^3 - 2x^2D^2 + 3xD + 7)y = x^3D^3y - 2x^2D^2y + 3xDy + 7y$. In general, we define an nth-order linear differential operator to be

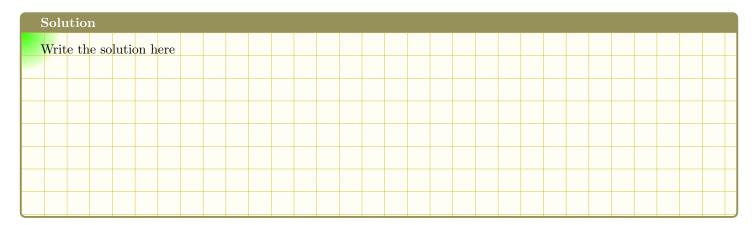
$$L = a_n(x)D^n + a_{n-1}(x)D^{n-1} + \ldots + a_1(x)D + a_0(x).$$

So, the generic form of a linear differential equation of order n can be rewritten as

Ly = g(x).

Example 6: Working with differential operators





Example 7: Factor a differential operator

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Solution	
Write the solution here	