Reduction of Order

Reduction of Order Method

Consider the linear, second-order, homogeneous differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0.$$

By the result in Lecture 7, the general solution to this equation is $y = c_1y_1 + c_2y_2$ where y_1 and y_2 are linearly independent solutions of the equation on some interval I. In the next lecture, we will learn a technique to relate this equation to an algebraic equation. The technique will help us find the two independent solutions y_1 and y_2 in many cases but in some cases it only gives a single solution. To find the second solution, we can apply the method of **reduction of order**.

Reduction of Order Method: Assume that we have found a nontrivial solution y_1 of the homogeneous equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0.$$

We seek a second solution y_2 such that y_1 and y_2 are linearly independent. Recall that *independence* means y_2 is not a *constant multiple* of y_1. So we assume that $y_2(x) = u(x)y_1(x)$ for some function $u(x)$. It follows that $y_2' = uy_1' + y_1u'$ and $y_2'' = uy_1'' + 2y_1'u' + y_1u''$. Substitute y_2, y_2' and y_2'' into the equation. Via a substitution $w = u'$, we will obtain a linear first order equation in w which we know how to solve. Next, we solve for w and then u. This helps us obtain y_2.

Example 1: Find a second solution by reduction of order - constant coefficients

Given that $y_1 = e^{5x}$ is a solution to $y'' - 25y = 0$ on the interval $(-\infty, \infty)$. Use reduction of order to find a second solution y_2. Form the general solution to the equation.

Solution

Write the solution here

Example 2: Find a second solution by reduction of order - nonconstant coefficients

Given that $y_1 = x$ is a solution to $x^2y'' - xy' + y = 0$ on the interval $(0, \infty)$. Use reduction of order to find a second solution y_2. Form the general solution to the equation.

Solution

Write the solution here

Example 3: A general equation

Suppose we divide the equation $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$ by $a_2(x)$ to obtain an equation in **standard form**

$$y'' + P(x)y' + Q(x)y = 0.$$

Assume that y_1 is a solution to the above equation. Use the reduction of order method to show that the second solution y_2 is given by

$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1^2} dx$$

Given that $y_1 = \sin(3x)$ is a solution of $y'' + 9y = 0$. Use the above formula to find the second solution y_2.

Solution

Write the solution here