Homogeneous Linear Equations with Constant Coefficients

Homogeneous Linear Second Order Equations

Consider the homogeneous linear second order equation with **constant coefficients**

$$ay'' + by' + cy = 0$$

where a, b and c are real constants and $a \neq 0$.
Method of solution:

1. Write down the associated **characteristic** (or **auxiliary**) equation, which is an algebraic equation of degree 2 in a variable m
$$am^2 + bm + c = 0.$$
2. The characteristic equation is a quadratic equation in m, its roots are
$$m_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } m_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$
   (i) If the discriminant $b^2 - 4ac$ is positive, then the roots m_1 and m_2 are real and distinct.
   (ii) If $b^2 - 4ac = 0$, the roots are real and equal, i.e., $m_1 = m_2$.
   (iii) If $b^2 - 4ac < 0$, the roots m_1 and m_2 are complex (non-real) and they are conjugate. In other words, $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$ where α and β are real and i is the *imaginary unit* with the property that $i^2 = -1$.

- **Case 1 - Distinct Real Roots**: The general solution to the equation is
$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x},$$
where c_1 and c_2 are arbitrary constants.
- **Case 2 - Repeated Real Roots:** In this case $m_1 = m_2$, the general solution is
$$y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}.$$
- **Case 3 - Conjugate Complex Roots:** In this case $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$, the general solution is
$$y = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x) = e^{\alpha x} \left(c_1 \cos(\beta x) + c_2 \sin(\beta x)\right).$$

Note 1: the above method of solution is based on the guess that a possible solution of the equation has the form $y = e^{mx}$. We now ask the question: for what values of m will $y = e^{mx}$ be a solution of the equation? Take the derivatives and substitute $y = e^{mx}$, $y' = me^{mx}$ and $y'' = m^2 e^{mx}$ into the equation, we obtain

$$am^2 e^{mx} + bme^{mx} + ce^{mx} = \underbrace{e^{mx}}_{\text{always } > 0} (am^2 + bm + c) = 0.$$

This gives rise to the characteristic equation.
Note 2: in the case of repeated real roots, the characteristic equation only gives us one solution $y = e^{m_1 x}$. The reduction of order method gives us the second solution $y = xe^{m_1 x}$.
Note 3: in the case of complex roots, we need Euler's formula $e^{i\beta x} = \cos(\beta x) + i \sin(\beta x)$ to eliminate the i and write the solution in terms of sine and cosine.

Example 1: Solve linear second order equations

Find the general solution to the given differential equation:

1. $y'' - 3y' + 2y = 0$
2. $y'' + 8y' + 16y = 0$
3. $y'' - 4y' + 7y = 0$

Solution

Write the solution here

Example 2: Solve an IVP

Solve the IVP: $y'' + 2y' + 17y = 0$; $y(0) = 1$, $y'(0) = -1$.

Solution

Write the solution here

Higher Order Equations

Consider the homogeneous linear nth-order equation with **constant coefficients**

$$a_n \frac{dy^n}{dx^n} + a_{n-1} \frac{dy^{n-1}}{dx^{n-1}} + \ldots + a_1 \frac{dy}{dx} + a_0 y = 0$$

where a_n, $a_{n-1}, \ldots, a_0$ are real constants and $a_n \neq 0$.
The associated characteristic equation is an algebraic equation of degree n in a variable m:

$$a_n m^n + a_{n-1} m^{n-1} + \ldots + a_1 m + a_0 = 0.$$

In solving this equation, there are several scenarios that may occur:

1. All of its roots are real and distinct.
2. All of its roots are real but some of the roots are repeated.
3. Some or all of its roots are complex (non-real).
4. Some complex (non-real) roots are repeated

There could be other scenarios such as when some of the roots are real and some of the roots are non-real. However, if we know how to deal with the scenarios outlined above, we will be able to adapt the strategy to any other combinations.
Note: The complex (non-real) roots of the characteristic equations always appear in **conjugate pairs**. For example, if $2 + 3i$ is a root, then $2 - 3i$ must also be a root.

- **Case - Roots are real and distinct:** If the n roots of the characteristic equation are m_1, $m_2, \ldots, m_n$, then the general solution to the equation is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots + c_n e^{m_n x}.$$

- **Case - Roots are real but some are repeated:** If $m = m_1$ is a root which repeats k times, then it gives rise to k independent solutions

$$e^{m_1 x}, x e^{m_1 x}, x^2 e^{m_1 x}, \ldots, x^{k-1} e^{m_2 x},$$

so the general solution will contain a linear combination of these independent solutions:

$$e^{m_1 x}(c_1 + c_2 x + c_3 x^2 + \ldots + c_k x^{k-1}).$$

For example, if the characteristic equation is

$$(m-5)^2 (m+2)^3 (m-7)(m+4) = 0,$$

then the general solution will be

$$y = e^{5x}(c_1 + c_2 x) + e^{-2x}(c_3 + c_4 x + c_5 x^2) + c_6 e^{7x} + c_7 e^{-4x}.$$

- **Case - Complex roots:** Any complex (non-real) roots must occur in conjugate pairs. Hence, if $\alpha + i\beta$ is a root, then $\alpha - i\beta$ must also be a root. This pair of complex roots give rise to two independent solutions

$$e^{\alpha x} \cos(\beta x), e^{\alpha x} \sin(\beta x),$$

so the general solution will contain a linear combination of these two independent solutions:

$$e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x)).$$

- **Case - Repeated complex roots:** If $\alpha + i\beta$ is a complex root which repeats k times, then $\alpha - i\beta$ must be a complex root which repeats k times. These roots give rise to the $2k$ independent solutions

$$e^{\alpha x} \cos(\beta x), x e^{\alpha x} \cos(\beta x), \ldots, x^{k-1} e^{\alpha x} \cos(\beta x)$$

$$e^{\alpha x} \sin(\beta x), x e^{\alpha x} \sin(\beta x), \ldots, x^{k-1} e^{\alpha x} \sin(\beta x)$$

so the general solution will contain a linear combination of all of these independent solutions.

Example 3: Real and Distinct Roots

Find the general solution to the equation $y''' + 3y'' - 4y' - 12y = 0$.

Solution

Write the solution here

Example 4: Repeated Real Roots

Find the general solution to the equation $y^{(4)} - 8y'' + 16y = 0$.

Solution

Write the solution here

Example 5: Distinct Complex Roots

Find the general solution to the equation $y^{(4)} + 5y'' + 6y = 0$.

Solution

Write the solution here

Example 6: Repeated Complex Roots

Find the general solution to the equation $y^{(4)} + 4y'' + 4y = 0$.

Solution

Write the solution here