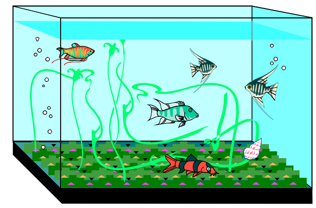
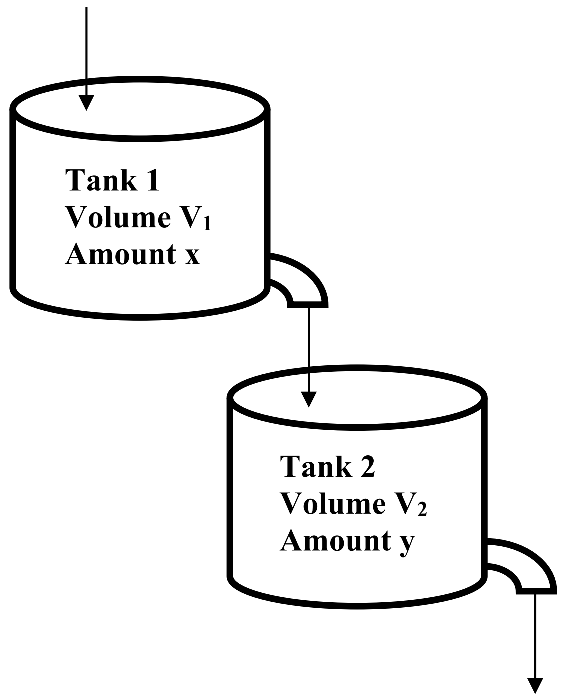
Math 2320 Application Problems



3. A tank initially contains 60 gallons of pure water. Salt water containing 1 pound of salt per gallon enters the tank at 2 gallons per minute, and the well-mixed solution leaves the tank at 3 gallons per minute; thus the tank is empty after exactly 1 hour.

a) Find a formula for the amount of water in the tank after t minutes.

b) What are the maximum and minimum amounts of salt in the tank?



4. Consider the cascade of 2 tanks shown in the figure, with $V_1 = 100$ gallons and $V_2 = 200$ gallons being the volumes of saltwater in the two tanks. Each tank initially contains 50 pounds of salt. The three flow rates are each 5 gallons per second, with pure water flowing into tank 1.

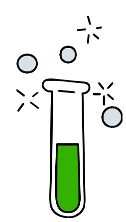
a) Find the amount $x(t)$ of salt in tank 1 at time t.

b) Suppose that $y(t)$ is the amount of salt in tank 2 at time t. Show first that

$$\frac{dy}{dt} = \frac{5x}{100} - \frac{5y}{200},$$

and then solve for $y(t)$ using the function $x(t)$ found in part a).

c) Find the maximum and minimum amounts of salt in tank 2.

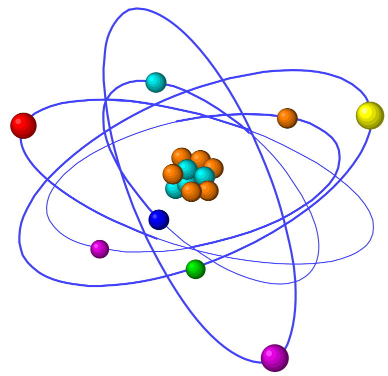


5. In a chemical reaction a substance S_1 with initial concentration K is transformed into a substance S_2. Let $y(t)$ be the concentration of S_2 at time t, so that $K - y(t)$ is the concentration of S_1 at time t. If the reaction is autocatalytic, the reaction is stimulated by S_2, and then $\frac{dy}{dt}$ is proportional to y and $K - y$; thus

$$\frac{dy}{dt} = ay(K - y)$$

for some positive reaction rate constant a. If the reaction is started at time $t = 0$ by introducing an initial concentration A of S_2, find the concentration of S_2 at time t.

6. In a second–order chemical reaction, a molecule of a substance S_1 and a molecule of a substance S_2 interact to produce a molecule of a new substance S_3. Suppose that the substances S_1 and S_2 have initial concentrations a and b, respectively, and let $y(t)$ be the concentration of S_3 at time t. Then at time t the concentrations of S_1 and S_2 are $a-y(t)$ and $b-y(t)$, respectively. The rate at which the reaction occurs is described by the ODE:



$$\frac{dy}{dt}=\alpha(a-y)(b-y)$$

where α is a positive reaction rate constant. If $y(0)=0$, find the concentration of y as a function of t in the following two cases:

a) $a \neq b$ **b)** $a = b$

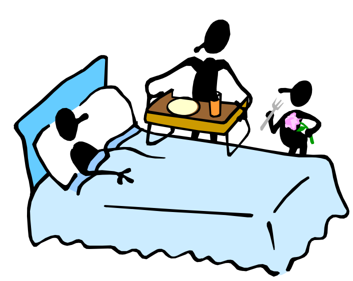
7. Consider a population of fixed size k. Let $y(t)$ the number of members of the population afflicted with a certain incurable(but not fatal) disease at time t. Suppose that none of the members of the population are immune and that the rate at which healthy members are infected is $by(t)$ per healthy member in unit time for some constant b. Then $y'(t)$ is the rate at which members are infected, and $y'=by(k-y)$. If the number of infectives at time $t=0$ is $A(0<A<k)$, then $y(0)=A$.

a) Solve the initial value problem $y'=by(k-y), y(0)=A$.

b) Find $\lim_{t\to\infty} y(t)$, and interpret this as to the ultimate fate of members of the population.

c) For what value of t will half of the population be infected?

8. Consider a disease spread by carriers who transmit the disease without exhibiting the symptoms themselves. Let y be the number of carriers and z the number of susceptibles at time t, and suppose that carriers are removed at a constant positive per capita rate α; then



$$\frac{dy}{dt}=-\alpha y.$$

Suppose that the disease spreads at a rate proportional to the product of y and z, so that:

$$\frac{dz}{dt}=-\beta yz$$

for some positive constant β. Let y_0 and z_0 be the initial number of carriers and susceptibles, respectively, at time $t = 0$.

a) Determine the number of carriers as a function of t; that is solve for y.

b) Substitute the solution $y(t)$ into the differential equation for z and solve to find the number of susceptibles as a function of t.

c) Find the limiting number of susceptibles as $t \to \infty$ (This is the number of members of the population who escape the disease.)

9. The population model $y' = by, y(0) = A$ does not allow for migration. A very crude model of a population with migration is given by

$$y' = by + M(t), y(0) = A$$

where $M(t)$ is the rate at which members are added to the population (if for some value of t, $M(t) > 0$, we have immigration to the population; whereas if $M(t) < 0$, we have emigration from the population.)

a) Solve the initial value problem if $b < 0$ and $M(t) = M_0 > 0$ are given constants, and find the limiting population as $t \to \infty$.

b) Solve the initial value problem if $b = -3$, $M(t) = 1 + m_0 \sin t$, where m_0 is a constant, and $A = 1000$. (Note that if $|m_0| > 1$, then $M(t)$ takes on both positive and negative values, indicating that there are periods of immigration and periods of emigration.) What happens as $t \to \infty$?

10. The U. S. Census population figures(in millions) for selected years from 1790 to 1970 are given in the table.

Year	**Population**
1790	3.93
1820	9.64
1850	23.19
1880	50.19
1910	92.23
1940	132.16
1970	203.30

a) Use this data to construct a model of the form

$$\frac{dP}{dt} = kP, P(0) = P_0$$

b) Construct a table comparing the population predicted by the model in part a) with the census populations.

c) Using the census data from 1790, 1850, and 1910 construct a population model of the form

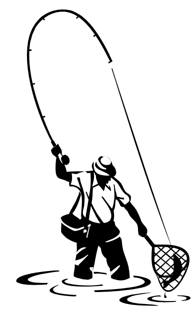
$$\frac{dP}{dt} = P(a - bP), P(0) = P_0.$$

d) Construct a table comparing the population predicted by the model in part c) with the census populations.

11. If a constant number h of animals is removed or harvested per unit time then a model of the population $P(t)$ of animals at time t is given by

$$\frac{dP}{dt} = P(a - bP) - h, P(0) = P_0,$$

where a, b, h, and P_0 are positive constants.



a) Solve the problem when $a = 5, b = 1,$ and $h = 4$.

b) Determine the long-term behavior of the population in part a) in the cases $P_0 > 4$, $1 < P_0 < 4$, and $0 < P_0 < 1$.

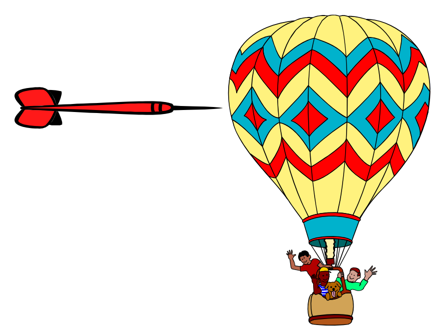
c) If the population becomes extinct in a finite time, find that time.

12. A differential equation governing the velocity v of a falling mass m subjected to air resistance proportional to the instantaneous velocity is

$$m\frac{dv}{dt} = mg - kv,$$

where k is a positive constant of proportionality.

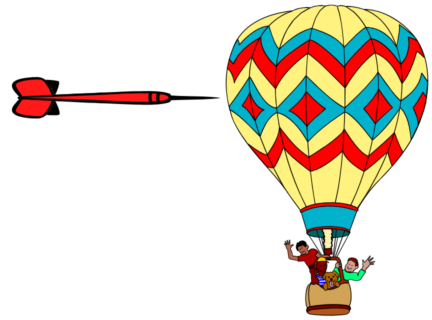
a) Solve this equation subject to the initial condition $v(0) = v_0$.



b) Determine the limiting or terminal velocity of the mass.

c) If distance s is related to velocity by $\frac{ds}{dt} = v$, find an explicit expression for s if it is further known that $s(0) = s_0$.

13. A differential equation governing the velocity v of a falling mass m subjected to air resistance proportional to the square of the instantaneous velocity is



$$m\frac{dv}{dt} = mg - kv^2,$$

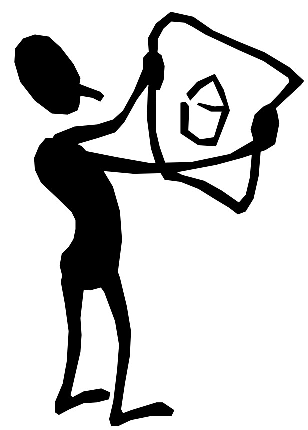
where k is a positive constant of proportionality.

a) Solve this equation subject to the initial condition $v(0) = v_0$.

b) Determine the limiting or terminal velocity of the mass.

c) If distance s is related to velocity by $\frac{ds}{dt} = v$, find an explicit expression for s if it is further known that $s(0) = s_0$.

14. When forgetfulness is taken into account, the rate of memorization of a subject is given by



$$\frac{dA}{dt} = k_1(M - A) - k_2 A,$$

where $k_1, k_2 > 0$, $A(t)$ is the amount of material memorized in time t, M is the total amount to be memorized, and $M - A$ is the amount remaining to be memorized.

a) Solve for $A(t)$, and graph the solution assuming that $A(0) = 0$.

b) Find the limiting value of A as $t \to \infty$, and interpret the result.

15. The populations of two species of animals are described by the nonlinear system of first order differential equations

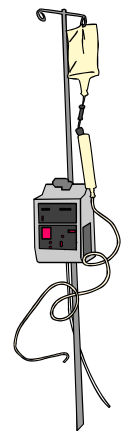
$$\frac{dx}{dt} = c_1 x(\alpha - x)$$
$$\frac{dy}{dt} = c_2 xy$$
,

where c_1, c_2 are positive constants, and α is the maximal sustainable population level of population x.

a) Solve for x and y in terms of t.

b) Plot the parametric curve $(x(t), y(t))$ for particular values of the constants.

20. One method of administering a drug is to feed it continuously into the bloodstream by a process called intravenous infusion. This may be modeled by the linear differential equation



$$\frac{dc}{dt} = -\mu c + D$$

where c is the concentration in the blood at time t, μ is a positive constant, and D is also a positive constant which is the rate at which the drug is administered.

a) Find the constant (or equilibrium) solution of the differential equation.

b) Given $c(0) = c_0$, find the concentration at time t.

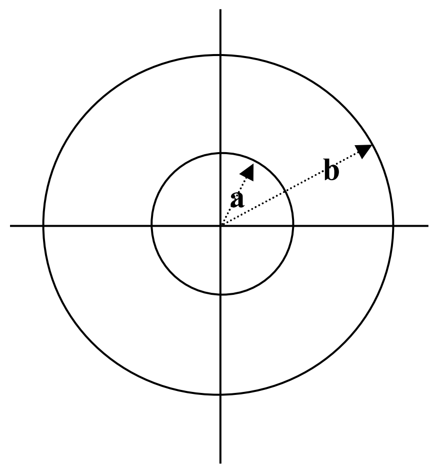
c) What limit does the concentration approach as $t \to \infty$?

d) Sketch the graph of a typical solution.

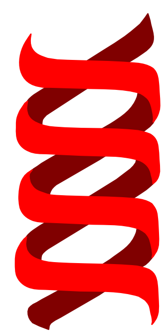
22. The temperature $u(r)$ in the circular ring shown in the figure is determined from the boundary-value problem

$$r\frac{d^2u}{dr^2} + \frac{du}{dr} = 0, u(a) = u_0, u(b) = u_1,$$

where u_0 and u_1 are constants. Solve for $u(r)$.



23. Consider one locus with two alleles, A_1 and A_2, in a randomly mating diploid population. That is, each individual in the population is either of type $A_1A_1, A_1A_2, or\ A_2A_2$. Let $p(t)$ be the frequency of the A_1 allele and $q(t)$ the frequency of the A_2 allele in the population at time t. Note that $p(t)+q(t)=1$. Denote the fitness of the A_iA_j by w_{ij}, and assume that $w_{11} = 1$, $w_{12} = 1-\frac{s}{2}$, and $w_{22} = 1-s$, where s is a nonnegative constant less than or equal to 1. That is, the fitness of the heterozygote A_1A_2 is halfway between the fitness of the other two homozygotes, and the type A_1A_1 is the fittest. If s is small, it can be shown that approximately



$$\frac{dp}{dt} = \tfrac{1}{2}sp(1-p) \quad \text{with} \quad p(0) = p_0$$

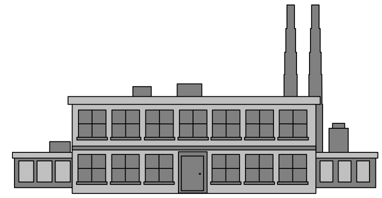
a) Solve this initial value problem.

b) Suppose that $p_0 = .1$ and $s = .01$, how long will it take until $p(t) = .5$?

c) Find $\lim_{t \to \infty} p(t)$. Explain in words what this limit means.

25. Under normal atmospheric conditions, the density of soot particles, $N(t)$, satisfies the differential equation

$$\frac{dN}{dt} = -k_c N^2 + k_d N$$



where k_c, called the coagulation constant, relates how well particles stick together and k_d, called the dissociation constant, relates how well particles fall apart. Both of these constants depend on temperature, pressure, particle size, and other external forces.

a) Solve $\frac{dN}{dt} = -k_c N^2 + k_d N$ for $N(t)$, of course, including an arbitrary constant C.

b) Find the constant C so that $N(t)$ satisfies the condition $N(0) = N_0$.

c) For each pair of values in the following table, sketch the graph of $N(t)$ if $N(0) = N_0$ for $N_0 = .01, .05, .1, .5, .75, 1, 1.5,$ *and,* 2. Regardless of the value of N_0, what do you notice in each case?

k_d	k_c
5	163
26	125
85	49

d) If $k_d > 0$, find $\lim_{t \to \infty} N(t)$.

26. When you are injured or sick, your doctor may prescribe antibiotics to prevent or cure infections. In the journal article "Changes in the Protein Profile of *Streptomyces Griseus* during a Cycloheximide Fermentation" we see that the production of the antibiotic cycloheximide by *Streptomyces* is typical of antibiotic production. During the production of cycloheximide, the mass of Streptomyces grows relatively quickly and produces little cycloheximide. After approximately 24 hours, the mass of Streptomyces remains relatively constant and cycloheximide accumulates. However, once the amount of cycloheximide reaches a certain level, extracellular cycloheximide is degraded (**feedback inhibited**). One approach to alleviating this problem and to maximize cycloheximide production is to continuously remove extracellular cycloheximide. The rate of growth of streptomyces can be described by the equation

$$\frac{dX}{dt} = \mu_{\max}\left(1 - \frac{X}{X_{\max}}\right)X$$

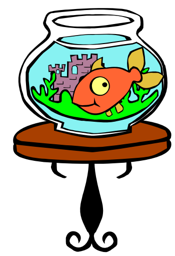
where X represents the mass concentration in g/L, μ_{max} is the maximum specific growth rate, and $X_{\max}$ represents the maximum mass concentration.

a) Solve the initial-value problem

$$\begin{cases} \dfrac{dX}{dt} = \mu_{\max}\left(1 - \dfrac{X}{X_{\max}}\right)X \\ X(0) = 1 \end{cases}.$$

b) Experimental results have shown that $\mu_{\max} = .3hr^{-1}$ and $X_{\max} = 10g/L$. Substitute these value into the result obtained in part a) and graph $X(t)$ on the interval $[0,24]$.

c) Find the mass concentration at the end of 4, 8, 12, 16, 20, and 24 hours.



27. When wading in a river or stream, you may notice that microorganisms like algae are frequently found on the rocks. Similarly if you have a swimming pool, you may notice that without maintaining appropriate levels of chlorine and algaecides, small patches of algae take over the pool surface, sometimes overnight. Underwater surfaces are attractive environments for microorganisms because water movement removes wastes and supplies a continuous supply of nutrients. On the other hand the organisms must spread over the surface without being washed away. If conditions become unfavorable, they must be able to free themselves from the surface and recolonize on a new surface. The rate at which microorganisms accumulate on a surface is proportional to the rate of growth of he microorganisms and the rate at which the microorganisms attach to the surface. An equation describing this situation is given by

$$\frac{dN}{dt} = \mu(N + A),$$

where N represents the microorganism density, μ the growth rate, A the attachment rate, ant t time.

a) If the attachment rate, A, is constant, solve the initial-value problem

$$\begin{cases} \dfrac{dN}{dt} = \mu(N + A) \\ N(0) = 0 \end{cases}$$

for N, and then solve for μ.

b) In a colony of microorganisms it was observed that $A = 3$. The number of microorganisms, N, at the end of t hours is shown in the following table. estimate the growth rate at the end of each hour.

t	**N**	μ
1	3	
2	9	

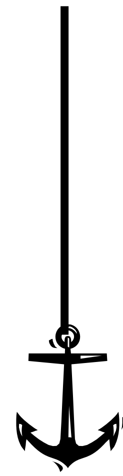
3	21	
4	45	

c) Using the growth rate obtained in part b), estimate the number of microorganisms at the end of 24 hours and 36 hours.

28. The differential equation that models the torsional motion of a weight suspended from an elastic shaft is

$$I\frac{d^2\theta}{dt^2}+c\frac{d\theta}{dt}+k\theta=T(t),$$

where θ represents the amount that the weight is twisted at time t, I is the moment of inertia of the weight, c is the damping constant, k is the elastic shaft constant, and $T(t)$ is the applied torque. Consider the differential equation with $I=1, c=4,\ and\ k=13$. Find $\theta(t)$ if



a) $T(t)=0, \theta(0)=\theta_0$, and $\frac{d\theta}{dt}(0)=0$.

b) $T(t)=\sin \pi t, \theta(0)=\theta_0$, and $\frac{d\theta}{dt}(0)=0$.

c) Describe the motion in both cases.

31. In a simplified model of a pair of guerilla forces in combat, the following system arises:

$$\frac{dy}{dt}=-axy$$

$$\frac{dx}{dt}=-bxy$$

where $x(t)$ and $y(t)$ are the strengths of opposing forces at time t and a and b are positive constants. The terms $-axy$ and $-bxy$ represent the combat loss rate for the troops *y* and *x*, respectively. This model assumes no reinforcements.

a) Show that *x* and *y* satisfy the equation

$$\frac{dy}{dx}=\frac{a}{b}.$$

b) Let $y(0)=y_0$ and $x(0)=x_0$. Solve the equation in part a) to find a relationship between *x* and *y*.

c) Determine which side wins in the cases of $\frac{y_0}{x_0}>\frac{a}{b}$ and $\frac{y_0}{x_0}<\frac{a}{b}$.

32. In a simplified model of a conflict between a guerilla force and a conventional force, the following system arises:

$$\frac{dy}{dt} = -ax$$
$$\frac{dx}{dt} = -bxy$$

where $x(t)$ is the strength of the guerilla force at time t, $y(t)$ is the strength of the conventional force at time t, and a and b are positive constants. The terms $-ax$ and $-bxy$ represent the combat loss rate for troops y and x, respectively. This model assumes no reinforcements.

a) Show that x and y satisfy the equation

$$\frac{dy}{dx} = \frac{a}{by}.$$
$$x(0) = x_0$$

b) Let $y(0) = y_0$ and. Solve the equation in part a) to find a relationship between x and y.

c) Determine which side wins in the cases of $\frac{y_0^2}{x_0} > \frac{2a}{b}$ and $\frac{y_0^2}{x_0} < \frac{2a}{b}$.

39. Groundwater containing an unknown (constant) concentration of pollutants, α, seeps at an unknown (constant) rate, r, into a cistern containing 1,000 gallons, and the well-stirred mixture leaks out at the same rate. Measurements show that the initial concentration of the pollutants is 1%. After one day the concentration is 1.1%, and after two days it is 1.19%. What is the concentration of the pollutants in the groundwater, and at what rate is the groundwater seeping into the cistern?

If x represents the amount of pollutant in the cistern, then the relevant differential equation is

$$\frac{dx}{dt} = \alpha r - \frac{rx}{1000}.$$

The given pollutant concentrations correspond to $x(0)=10$, $x(1)=11$, and $x(2)=11.9$. So use this information to solve for α and r.

41. The equations

$$\frac{dT^*}{dt} = kV_1T_0 - \delta T^*$$
$$\frac{dV_1}{dt} = -cV_1$$

are used in modeling HIV-1 infections. Here $T^* = T^*(t)$ denotes the number of infected cells, $T_0 = T(0)$ is the number of potentially infected cells at the time therapy is begun, $V_1 = V_1(t)$ is the concentration of viral particles in the plasma, k is the rate of infection, c is the rate constant for viral particle clearance, and δ is the rate of loss of virus producing cells.

a) Solve the system by first solving the second equation.

b) What happens to the number of infected cells, T*, as $t \to \infty$?

42. In trying to determine the shape of a flexible, nonstretching cable suspended between two points A and B of equal height, we can analyze the forces acting on the cable to get the differential equation

$$\frac{d^2y}{dx^2} = k\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}},$$

where $k > 0$ is a constant.

a) Use the substitution $u = \frac{dy}{dx}$ to reduce the DE to a separable first order equation.

b) Solve the DE in part a) and use it to solve the original DE.

43. In trying to regulate fishing in the oceans, international commissions have been set up to implement controls. To understand the effect of such controls, mathematical models of fish populations have been constructed. One stage in this modeling effort involves predicting the growth of an individual fish:

$$\frac{dW}{dt} = \alpha W^{\frac{2}{3}} - \beta W,$$

where $W = W(t)$ denotes the weight of a fish and α and β are positive constants.

a) Find the general solution of the equation.

b) Calculate $W_\infty = \lim_{t\to\infty} W(t)$, the limiting weight of the fish.

c) Find the particular solution if $W(0) = 0$.

d) Sketch the graph of W versus t.

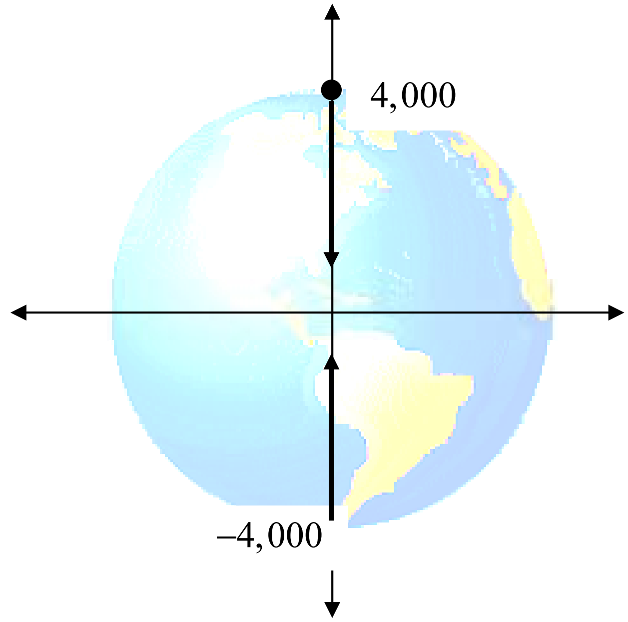
44. The equation $\theta'' = -4\theta - 5\theta'$ where $\theta = \theta(t)$ represents the angle made by a swinging door from its closed position. The initial conditions are $\theta(0) = \frac{\pi}{3}$ and $\theta'(0) = 0$.

a) Determine the angle $\theta(t)$ for $t > 0$.

b) What happens to this angle as t becomes large?

c) Graph the solution $\theta(t)$ on the interval $[0,5]$.

50. Inside the Earth, it can be shown that the force of gravity is proportional to the distance to the center of the Earth. Suppose that a hole is drilled through the Earth from pole to pole, and a rock is dropped into the hole. Neglecting air resistance, and assuming that the Earth is a perfect sphere of radius 4,000 miles



Let $y(t)$ represent the position of the object on the y-axis at time t. The initial conditions are $y(0)=4{,}000$ and $y'(0)=0$. Supposing that the constant of proportionality between the force and the distance from the center of the Earth is $g=\frac{32}{5280}=\frac{1}{165}\ {}^{miles}\!/_{\sec^2}$. So we need to solve

$$my''(t)=\frac{-mgy(t)}{R},$$

$y(0)=R$, $y'(0)=0$

Simplifying, we get that

$$y''(t)=-\frac{1}{165\cdot 4000}y(t),\ y(0)=4000,\ y'(0)=0$$

a) Find the time required for an object dropped into the hole to return.

b) Find the velocity of the object as it passes through the center of the Earth.

52. A differential equation that arises in the study of traffic flow is

$$\frac{dx}{dt}=\frac{1}{2}V+\frac{x}{2t},$$

where V is the maximum velocity of a car in traffic flow and x is the directed distance of a car from a traffic light. If the car starts from rest, then from the differential equation we have that $x=-x_0$ at $t=\frac{x_0}{V}$.

a) Solve the initial value problem, and show that V is in fact the maximum velocity of a car.

b) Find the time it takes the car to reach the traffic light, and compare it with the time it would take if the car were going at its maximum speed the entire distance from its starting place to the traffic light.