3.4. Zeros of Polynomial Functions

Thursday, October 31, 2019 9:40 AM

Obj 1: The Rational Zero Theorem

The Rational Zero Theorem:

If $f(x)$ is a polynomial with integer coefficients, then we can make a list of all possible rational zeros of $f(x)$ and we have

$$\text{possible rational zeros} = \frac{\text{Factors of constant term}}{\text{Factors of leading coefficient}}$$

E.g. List all the possible rational zeros of

$$f(x) = 4x^5 + 12x^4 - x - 3.$$

Constant term $= -3 \longrightarrow$ Factors: ± 1; ± 3

Leading coefficient $= 4 \longrightarrow$ Factors: ± 1; ± 2; ± 4

$$\text{Possible rational zeros} = \frac{\pm 1 \ ; \ \pm 3}{\pm 1 \ ; \ \pm 2 \ ; \ \pm 4}$$

$$= \left\{ \pm 1 \ ; \ \pm \frac{1}{2} \ ; \ \pm \frac{1}{4} \ ; \ \pm 3 \ ; \ \pm \frac{3}{2} \ ; \ \pm \frac{3}{4} \right\}$$

The Rational Zero Theorem says that if $f(x)$ has a rational zero (root), it must be one of the 12 numbers in this list.

E.g. List all the possible rational zeros of:

$$f(x) = 15x^3 + 2x^2 - 5x - 6$$

Constant term $= -6 \rightarrow$ Factors: $\pm 1; \pm 2; \pm 3; \pm 6$

Leading coeff. $= 15 \rightarrow$ Factors: $\pm 1; \pm 3; \pm 5; \pm 15$

Possible rational zeros $= \dfrac{\pm 1; \pm 2; \pm 3; \pm 6}{\pm 1; \pm 3; \pm 5; \pm 15}$

$$= \left\{ \pm 1; \pm \frac{1}{3}; \pm \frac{1}{5}; \pm \frac{1}{15}; \pm 2; \pm \frac{2}{3}; \pm \frac{2}{5}; \pm \frac{2}{15}; \pm 3; \pm \frac{3}{5}; \pm 6; \pm \frac{6}{5} \right\}$$

Obj 2: Find zeros of a polynomial function using the Rational Zero Theorem and synthetic division.

E.g. Find all the zeros (roots) of:

$$f(x) = x^3 + 7x^2 + 11x - 3.$$

Step 1: Use the Rational Zero Theorem to make a list of possible rational zeros.

Constant term $= -3 \rightarrow$ Factors: $\pm 1; \pm 3$

Leading coeff $= 1 \rightarrow$ Factors: ± 1

Possible rational zeros $= \dfrac{\pm 1, \pm 3}{\pm 1} = \{\pm 1; \pm 3\}$

Step 2: Test which number in the list is a zero by synthetic division.

* Test 1

$$\begin{array}{r|rrrr} 1 & 1 & 7 & 11 & -3 \\ & & 1 & 8 & 19 \\ \hline & 1 & 8 & 19 & \boxed{16} \end{array} \rightarrow \text{Remainder} = 16$$

→ Not a zero

→ continue testing

* Test 3

$$\begin{array}{r|rrrr} 3 & 1 & 7 & 11 & -3 \\ & & 3 & 30 & 123 \\ \hline & 1 & 10 & 41 & \boxed{120} \end{array} \rightarrow R \neq 0$$

* Test −1

$$\begin{array}{r|rrrr} -1 & 1 & 7 & 11 & -3 \\ & & -1 & -6 & -5 \\ \hline & 1 & 6 & 5 & \boxed{-8} \end{array} \rightarrow R \neq 0$$

* Test −3

$$\begin{array}{r|rrrr} -3 & 1 & 7 & 11 & -3 \\ & & -3 & -12 & 3 \\ \hline & 1 & 4 & -1 & \boxed{0} \end{array} \rightarrow R = 0$$

→ So, $x = -3$ is a zero of $f(x)$.

Quotient $= x^2 + 4x - 1$.

Step 3: Set quotient $= 0$ to find remaining zeros

$$x^2 + 4x - 1 = 0$$

Quadratic Formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$a = 1$; $b = 4$; $c = -1$

$$x = \frac{-4 \pm \sqrt{(4)^2 - 4(1)(-1)}}{2\cdot(1)}$$

$$x = \frac{-4 \pm \sqrt{20}}{2} = \frac{-4 \pm \sqrt{4\cdot 5}}{2} = \frac{-4 \pm 2\sqrt{5}}{2}$$

$$x = \frac{\cancel{2}(-2 \pm \sqrt{5})}{\cancel{2}} = -2 \pm \sqrt{5}.$$

Conclusion : the zeros of $f(x)$ are :

$$\boxed{x = -3; \quad x = -2 + \sqrt{5}; \quad x = -2 - \sqrt{5}}$$

E.g. Find all the zeros of $f(x) = x^3 + x^2 - 5x - 2$.

Step 1: List of possible rational zeros :

Constant term $= -2 \to$ Factors: $\pm 1, \pm 2$
Leading coeff $= 1 \to$ Factors: ± 1

Possible rational zeros $= \{\pm 1, \pm 2\}$

Step 2: Test

$$\begin{array}{r|rrrr} 2 & 1 & 1 & -5 & -2 \\ & & 2 & 6 & 2 \\ \hline & \boxed{1 \quad 3 \quad 1} & & & \boxed{0} \end{array}$$

$\boxed{0} \to R = 0$

$\to$ 2 is a zero.

$\to$ Quotient : $x^2 + 3x + 1$

Step 3: Set quotient $= 0$

$x^2 + 3x + 1 = 0$; $a = 1; b = 3; c = 1$