1.6. Other types of Equations

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Objective 1: Solve Polynomial Equations by Factoring.

E.g. $4x^4 = 12x^2$

Step 1: Get 0 on RHS:

$$4x^4 - 12x^2 = 0$$

Step 2: Factor

$$4x^2(x^2 - 3) = 0$$

Step 3: Set each factor equal to 0 and solve

$4x^2 = 0$ or $x^2 - 3 = 0$

$\rightarrow x^2 = \frac{0}{4}$ $\rightarrow x^2 = 3$

$\rightarrow x^2 = 0$ $\rightarrow \boxed{x = \pm\sqrt{3}}$

$\rightarrow \boxed{x = 0}$

Solution set: $\{0, \sqrt{3}, -\sqrt{3}\}$

E.g. Factor by grouping

$$x^3 + x^2 = 4x + 4$$

Step 1: Get zero on RHS.

$$\underbrace{x^3 + x^2}\ \underbrace{-4x - 4} = 0$$

Step 2: Factor by grouping

$$\underbrace{x^2\boxed{(x+1)}}_{\text{Factor out } x^2 \text{ from first 2 terms}} \underbrace{-4\boxed{(x+1)}}_{\text{Factor out } -4 \text{ from the last 2 terms}} = 0$$

$$\underbrace{(x+1)(x^2-4)}_{} = 0$$

Factor out the factor $x+1$ from the 2 groups

Step 3: Set each factor equal to zero and solve.

$x + 1 = 0$ or $x^2 - 4 = 0$

$\rightarrow x = -1$ $\rightarrow x^2 = 4$

$\rightarrow x = \pm\sqrt{4}$

$\rightarrow x = \pm 2$

Solution set: $\{-1; 2, -2\}$

E.g. Factor by grouping

$$2x^3 + 3x^2 = 8x + 12$$

$$\boxed{\underbrace{2x^3 + 3x^2}}\ \boxed{\underbrace{-8x - 12}} = 0 \quad (\text{Get } 0 \text{ on RHS})$$

Factor out x^2

Factor out -4

$$x^2\boxed{(2x+3)} - 4\boxed{(2x+3)} = 0$$

Factor out $(2x+3)$

$$(2x+3)(x^2-4) = 0$$

$2x + 3 = 0$ or $x^2 - 4 = 0$ (Set each factor equal to 0)

$\rightarrow x = -\frac{3}{2}$ $x^2 = 4$

$x = \pm\sqrt{4}$

$\rightarrow x = \pm 2$

Solution set: $\boxed{\left\{-\frac{3}{2}, 2, -2\right\}}$

Objective 2: Solving Radical Equations

E.g. $\sqrt{2x-1} + 2 = x$

Step 1: Isolate the Radical on one side:

$$\sqrt{2x-1} = x - 2$$

Step 2: Square both sides:

$$\left(\sqrt{2x-1}\right)^2 = (x-2)^2$$

$$2x - 1 = (x-2)^2$$

Note: squaring gets rid of the square root

Step 3: Solve the resulting equation

$$2x - 1 = (x-2)(x-2)$$

$$2x - 1 = x^2 \underbrace{-2x - 2x}_{} + 4$$

(combine like terms)

$$2x - 1 = x^2 - 4x + 4$$

$$0 = x^2 - 4x + 4 - 2x + 1$$

$$0 = x^2 - 6x + 5$$

$$0 = (x-1)(x-5)$$

$$x - 1 = 0 \quad \text{or} \quad x - 5 = 0$$

$$x = 1 \quad ; \quad x = 5$$

Step 5: Check your solutions:

Original equation: $\sqrt{2x-1} + 2 \stackrel{?}{=} x$

Check $x = 1$: $\sqrt{2(1)-1} + 2 \stackrel{?}{=} 1$

$$\sqrt{1} + 2 \stackrel{?}{=} 1$$
$$1 + 2 \stackrel{?}{=} 1 \quad \text{False.}$$

So, $x = 1$ is NOT a solution to the original equation.

Note: This is often called an extraneous solution.
(It is a solution to the resulting equation but NOT a solution to the original equation)

Check $x = 5$: $\sqrt{2(5) - 1} + 2 \stackrel{?}{=} 5$

$$\sqrt{9} + 2 \stackrel{?}{=} 5$$
$$3 + 2 \stackrel{?}{=} 5. \quad \text{True.}$$

So, $x = 5$ is a solution to the original equation.

Conclusion: Solution set to the original equation:

$$\boxed{\{5\}}$$

E.g. Solve: $\sqrt{x+3} + 3 = x$

$$\sqrt{x+3} = x - 3 \quad \text{(isolate the square root)}$$
$$\left(\sqrt{x+3}\right)^2 = (x-3)^2 \quad \text{(Square both sides)}$$
$$x + 3 = x^2 - 6x + 9$$
$$0 = x^2 - 7x + 6$$
$$0 = (x-1)(x-6)$$

$x - 1 = 0$; $x - 6 = 0$

$x = 1$; $x = 6$

Check our solutions:

Check $\boxed{x = 1}$: $\underbrace{\sqrt{(1)+3} + 3}_{5} \stackrel{?}{=} 1$

$5 \stackrel{?}{=} 1$ False.

Not a solution.

Check $\boxed{x = 6}$: $\underbrace{\sqrt{6+3} + 3}_{6} \stackrel{?}{=} 6$

is a solution $6 \stackrel{?}{=} 6$ True

Solution set: $\boxed{\{6\}}$

Objective: Solve Equations that are quadratic in Form by substitution method

E.g. Solve: $\boxed{x^4} - 8x^2 - 9 = 0$

Note: This is a polynomial equation. It is hard to do factor by grouping because we only have 3 terms.

Note: $x^4 = (x^2)^2$. This suggests that we can do a u-substitution.

$\boxed{\text{Let } u = x^2}$ (Substitution step)

Rewrite the equation with the variable u.

$u^2 - 8u - 9 = 0$

Solve the equation with the variable u.

$$u^2 - 8u - 9 = 0$$

$$(u+1)(u-9) = 0$$

$u + 1 = 0$ or $u - 9 = 0$

$u = -1$; $u = 9$.

Solve for the variable x.

Recall: $u = x^2$.

When $u = -1$, this means $x^2 = -1$

$$x = \pm\sqrt{-1} = \pm\sqrt{i^2} = \pm i$$

When $u = 9$; this mean: $x^2 = 9$

$$x = \pm\sqrt{9} = \pm 3.$$

Solution set: $\boxed{\{i, -i, 3, -3\}}$

E.g. $(x^2-4)^2 + (x^2-4) - 6 = 0$

Substitution: Let $x^2 - 4 = u$

Rewrite the equation in the variable u:

$$u^2 + u - 6 = 0$$

$$(u-2)(u+3) = 0$$

$u - 2 = 0$; $u + 3 = 0$

$u = 2$; $u = -3$

Solve for x. Recall: $u = x^2 - 4$

When $u = 2$, this means: $x^2 - 4 = 2$

$x^2 = 6 \longrightarrow x = \pm\sqrt{6}$

When $u = -3$, this means $x^2 - 4 = -3$

$x^2 = 1 \rightarrow x = \pm\sqrt{1} = \pm 1$

Solution set: $\{\sqrt{6}, -\sqrt{6}, 1, -1\}$

E.g. Solve $x^4 - 5x^2 + 6 = 0$

Let $u = x^2$.

Rewrite: $u^2 - 5u + 6 = 0$

$(u-2)(u-3) = 0$

$u - 2 = 0$ or $u - 3 = 0$

$u = 2$; $u = 3$

Solve for x.

$x^2 = 2$ or $x^2 = 3$

$x = \pm\sqrt{2}$ $x = \pm\sqrt{3}$

Solution set: $\boxed{\{\sqrt{2}, -\sqrt{2}, \sqrt{3}, -\sqrt{3}\}}$