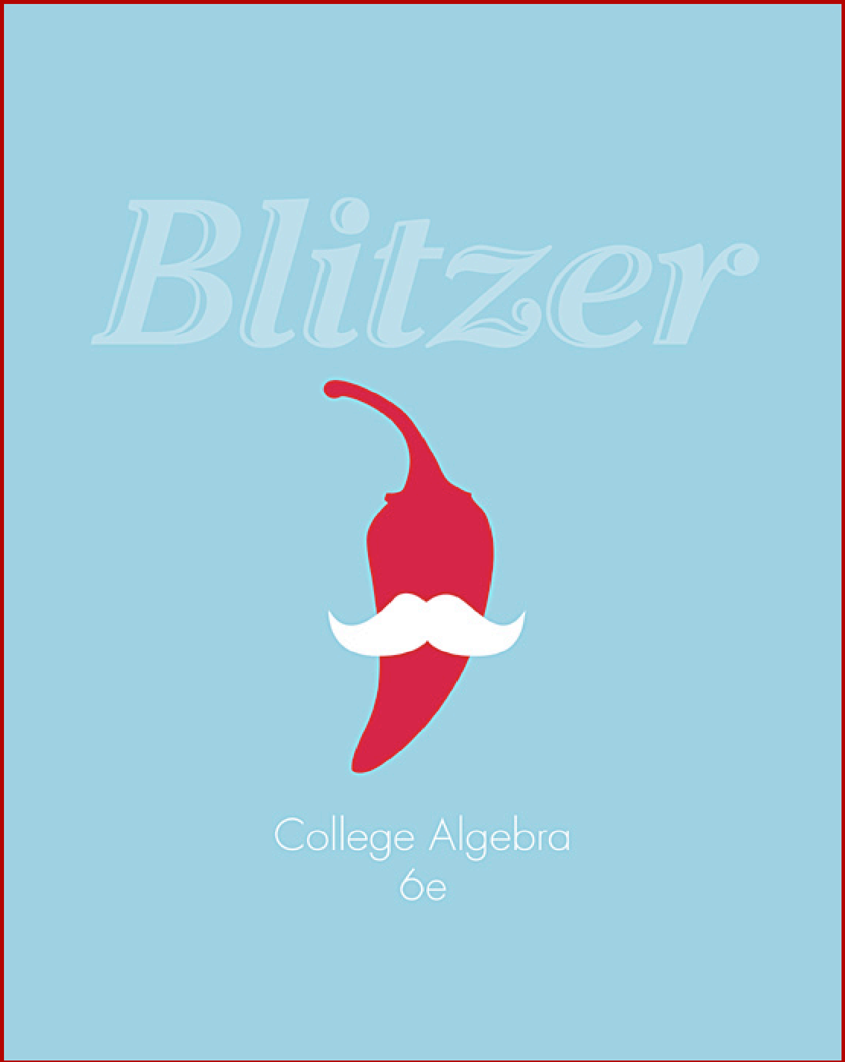
Chapter 6

Matrices and Determinants

6.1 Matrix Solutions to Linear Systems



Objectives:

- Write the augmented matrix for a linear system.
- Perform matrix row operations.
- Use matrices and Gaussian elimination to solve systems.
- Use matrices and Gauss-Jordan elimination to solve systems.

Matrices

Matrices are used to display information and to solve systems of linear equations. A **matrix** (plural: **matrices**) is a rectangular array. The information is arranged in rows and columns and placed in brackets. The values inside the brackets are called **elements** of the matrix.

matrix $\longrightarrow$ $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

elements of the matrix

Augmented Matrices

The first step in solving a system of linear equations using matrices is to write the *augmented matrix*. An **augmented matrix** has a vertical bar separating the columns of the matrix into two groups. The coefficients of each variable are placed to the left of the vertical line and the constants are placed to the right. If any variable is missing, its coefficient is 0.

Augmented Matrices *(continued)*

Here are two examples of augmented matrices:

System of Linear Equations	Augmented Matrix
$\begin{cases} 3x + y + 2z = 31 \\ x + y + 2z = 19 \\ x + 3y + 2z = 25 \end{cases}$	$\left[\begin{array}{ccc\|c} 3 & 1 & 2 & 31 \\ 1 & 1 & 2 & 19 \\ 1 & 3 & 2 & 25 \end{array}\right]$
$\begin{cases} x + 2y - 5z = -19 \\ y + 3z = 9 \\ z = 4 \end{cases}$	$\left[\begin{array}{ccc\|c} 1 & 2 & -5 & -19 \\ 0 & 1 & 3 & 9 \\ 0 & 0 & 1 & 4 \end{array}\right]$

Matrix Row Operations

A matrix with 1's shown on the main diagonal and 0's below the 1's is said to be in **row-echelon form**. We use **row operations** on the augmented matrix to produce a matrix in row-echelon form.

main diagonal

$$\left[\begin{array}{ccc|c} 1 & 2 & -5 & -19 \\ 0 & 1 & 3 & 9 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

This matrix is in row-echelon form.

Matrix Row Operations

Matrix Row Operations

The following row operations produce matrices that represent systems with the same solution set:

1. Two rows of a matrix may be interchanged. This is the same as interchanging two equations in a linear system.
2. The elements in any row may be multiplied by a nonzero number. This is the same as multiplying both sides of an equation by a nonzero number.
3. The elements in any row may be multiplied by a nonzero number, and these products may be added to the corresponding elements in any other row. This is the same as multiplying both sides of an equation by a nonzero number and then adding equations to eliminate a variable.

Two matrices are **row equivalent** if one can be obtained from the other by a sequence of row operations.

Symbols for Row Operations

Each matrix row operation in the preceding box can be expressed symbolically as follows:

1. Interchange the elements in the *i*th and *j*th rows: $R_i \leftrightarrow R_j$.

2. Multiply each element in the *i*th row by k: kR_i.

3. Add k times the elements in row i to the corresponding elements in row j: $kR_i + R_j$.

Example: Performing Matrix Row Operations

Use the matrix $\left[\begin{array}{ccc|c} 4 & 12 & -20 & 8 \\ 1 & 6 & -3 & 7 \\ -3 & -2 & 1 & -9 \end{array}\right]$

and perform the indicated row operation: $R_1 \leftrightarrow R_2$

$$\left[\begin{array}{ccc|c} 4 & 12 & -20 & 8 \\ 1 & 6 & -3 & 7 \\ -3 & -2 & 1 & -9 \end{array}\right] \quad R_1 \leftrightarrow R_2 \quad \left[\begin{array}{ccc|c} 1 & 6 & -3 & 7 \\ 4 & 12 & -20 & 8 \\ -3 & -2 & 1 & -9 \end{array}\right]$$

Example: Performing Matrix Row Operations

Use the matrix $\begin{bmatrix} 4 & 12 & -20 & | & 8 \\ 1 & 6 & -3 & | & 7 \\ -3 & -2 & 1 & | & -9 \end{bmatrix}$

and perform the indicated row operation: $\frac{1}{4}R_1$

$$\begin{bmatrix} \frac{1}{4}(4) & \frac{1}{4}(12) & \frac{1}{4}(-20) & | & \frac{1}{4}(8) \\ 1 & 6 & -3 & | & 7 \\ -3 & -2 & 1 & | & -9 \end{bmatrix} = \begin{bmatrix} 1 & 3 & -5 & | & 2 \\ 1 & 6 & -3 & | & 7 \\ -3 & -2 & 1 & | & -9 \end{bmatrix}$$

Example: Performing Matrix Row Operations

Use the matrix $\left[\begin{array}{ccc|c} 4 & 12 & -20 & 8 \\ 1 & 6 & -3 & 7 \\ -3 & -2 & 1 & -9 \end{array}\right]$ and perform the indicated row operation: $3R_2 + R_3$

$$\left[\begin{array}{ccc|c} 4 & 12 & -20 & 8 \\ 1 & 6 & -3 & 7 \\ 3(1)+(-3) & 3(6)+(-2) & 3(-3)+1 & 3(7)+(-9) \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 4 & 12 & -20 & 8 \\ 1 & 6 & -3 & 7 \\ 0 & 16 & -8 & 12 \end{array}\right]$$

Solving Linear Systems Using Gaussian Elimination

Solving Linear Systems of Three Equations with Three Variables Using Gaussian Elimination

1. Write the augmented matrix for the system.

2. Use matrix row operations to simplify the matrix to a row-equivalent matrix in row-echelon form,with 1s down the main diagonal from upper left to lower right, and 0s below the 1s in the first and second columns.

$$\left[\begin{array}{ccc|c} 1 & * & * & * \\ * & * & * & * \\ * & * & * & * \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & * & * & * \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & * & * \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{array}\right]$$

Get 1 in the upper left-hand corner.

Use the 1 in the first column to get 0s below it.

Get 1 in the second row, second column position.

Use the 1 in the second column to get 0 below it.

Get 1 in the third row, third column position.

3. Write the system of linear equations corresponding to the matrix in step 2 and use back-substitution to find the system's solution.

Example: Gaussian Elimination with Back-Substitution

Use matrices to solve the system:

$$\begin{cases} 2x + y + 2z = 18 \\ x - y + 2z = 9 \\ x + 2y - z = 6 \end{cases}$$

Step 1 Write the augmented matrix for the system.

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 18 \\ 1 & -1 & 2 & 9 \\ 1 & 2 & -1 & 6 \end{array}\right]$$

Example: Gaussian Elimination with Back-Substitution *(continued)*

Use matrices to solve the system:
$$\begin{cases} 2x + y + 2z = 18 \\ x - y + 2z = 9 \\ x + 2y - z = 6 \end{cases}$$

Step 2 Use matrix row operations to simplify the matrix to row-echelon form, with 1's down the main diagonal from upper left to lower right, and 0's below the 1's in the first and second columns.

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 18 \\ 1 & -1 & 2 & 9 \\ 1 & 2 & -1 & 6 \end{array}\right] R_1 \leftrightarrow R_2 \left[\begin{array}{ccc|c} 1 & -1 & 2 & 9 \\ 2 & 1 & 2 & 18 \\ 1 & 2 & -1 & 6 \end{array}\right]$$

Example: Gaussian Elimination with Back-Substitution *(continued)*

Use matrices to solve the system:
$$\begin{cases} 2x + y + 2z = 18 \\ x - y + 2z = 9 \\ x + 2y - z = 6 \end{cases}$$

Step 2 *(cont)* Use matrix row operations to simplify the matrix to row-echelon form.

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 9 \\ 2 & 1 & 2 & 18 \\ 1 & 2 & -1 & 6 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & -1 & 2 & 9 \\ 0 & 3 & -2 & 0 \\ 1 & 2 & -1 & 6 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & -1 & 2 & 9 \\ 0 & 3 & -2 & 0 \\ 0 & 3 & -3 & -3 \end{array}\right]$$

replace row 2 by $-2R_1 + R_2$

replace row 3 by $-1R_1 + R_3$

Example: Gaussian Elimination with Back-Substitution *(continued)*

Use matrices to solve the system:

$$\begin{cases} 2x + y + 2z = 18 \\ x - y + 2z = 9 \\ x + 2y - z = 6 \end{cases}$$

Step 2 *(cont)* Use matrix row operations to simplify the matrix to row-echelon form.

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 9 \\ 0 & 3 & -2 & 0 \\ 0 & 3 & -3 & -3 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & -1 & 2 & 9 \\ 0 & 3 & -3 & -3 \\ 0 & 3 & -2 & 0 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & -1 & 2 & 9 \\ 0 & 1 & -1 & -1 \\ 0 & 3 & -2 & 0 \end{array}\right]$$

$R_2 \leftrightarrow R_3$ replace row 2 by $\frac{1}{3}R_2$

Example: Gaussian Elimination with Back-Substitution *(continued)*

Use matrices to solve the system:
$$\begin{cases} 2x + y + 2z = 18 \\ x - y + 2z = 9 \\ x + 2y - z = 6 \end{cases}$$

Step 2 *(cont)* Use matrix row operations to simplify the matrix to row-echelon form.

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 9 \\ 0 & 1 & -1 & -1 \\ 0 & 3 & -2 & 0 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & -1 & 2 & 9 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

replace row 2 by
$-3R_2 + R_3$

Example: Gaussian Elimination with Back-Substitution *(continued)*

Use matrices to solve the system:
$$\begin{cases} 2x + y + 2z = 18 \\ x - y + 2z = 9 \\ x + 2y - z = 6 \end{cases}$$

Step 3 Write the system of linear equations corresponding to the matrix in step 2 and use back-substitution to find the system's solution.

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 9 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right] \qquad \begin{cases} x - y + 2z = 9 \\ y - z = -1 \\ z = 3 \end{cases}$$

Example: Gaussian Elimination with Back-Substitution *(continued)*

Use matrices to solve the system:

$$\begin{cases} 2x + y + 2z = 18 \\ x - y + 2z = 9 \\ x + 2y - z = 6 \end{cases}$$

Step 3 *(cont)* ...use back-substitution to find the system's solution.

$$\begin{cases} x - y + 2z = 9 \\ \qquad y - z = -1 \\ \qquad\quad z = 3 \end{cases}$$

$$\begin{aligned} y - z &= -1 \\ y - 3 &= -1 \\ y &= 2 \end{aligned}$$

$$\begin{aligned} x - y + 2z &= 9 \\ x - 2 + 2(3) &= 9 \\ x - 2 + 6 &= 9 \\ x + 4 &= 9 \\ x &= 5 \end{aligned}$$

The solution set is $\{(5, 2, 3)\}$.

Solving Linear Systems Using Gauss-Jordan Elimination

Solving Linear Systems Using Gauss-Jordan Elimination

1. Write the augmented matrix for the system.
2. Use matrix row operations to simplify the matrix to a row-equivalent matrix in reduced row-echelon form, with 1s down the main diagonal from upper left to lower right, and 0s above and below the 1s.
   - **a.** Get 1 in the upper left-hand corner.
   - **b.** Use the 1 in the first column to get 0s below it.
   - **c.** Get 1 in the second row, second column.
   - **d.** Use the 1 in the second column to make the remaining entries in the second column 0.
   - **e.** Get 1 in the third row, third column.
   - **f.** Use the 1 in the third column to make the remaining entries in the third column 0.
   - **g.** Continue this procedure as far as possible.
3. Use the reduced row-echelon form of the matrix in step 2 to write the system's solution set. (Back-substitution is not necessary.)

Example: Using Gauss-Jordan Elimination

Solve the following system using Gauss-Jordan elimination.

$$\begin{cases} 2x + y + 2z = 18 \\ x - y + 2z = 9 \\ x + 2y - z = 6 \end{cases}$$

When we solved this system using Gaussian elimination with back-substitution, we simplified the system to this matrix:

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 9 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

We will begin our work from this step.

Example: Using Gauss-Jordan Elimination *(continued)*

Solve the following system using Gauss-Jordan elimination.

$$\begin{cases} 2x + y + 2z = 18 \\ x - y + 2z = 9 \\ x + 2y - z = 6 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 9 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 0 & 1 & 8 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

replace row 1 by $R_1 + R_2$

replace row 1 by $-R_3 + R_1$

Example: Using Gauss-Jordan Elimination *(continued)*

Solve the following system using Gauss-Jordan elimination.

$$\begin{cases} 2x + y + 2z = 18 \\ x - y + 2z = 9 \\ x + 2y - z = 6 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

replace row 2 by

$R_3 + R_2$

This last matrix corresponds to $x = 5, y = 2, z = 3.$ The solution set is $\{(5,2,3)\}$.